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THE ELEMENTS

SPHERICAL TRIGONOMETRY

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ERRATA TO THE *PLANE* TRIGONOMETRY.

Page 10, Line 5, *for* AP *read* AP^2.
„ 9, *for* DP *read* DP^2.
53, „ 3 from top, *for* sin 2 A *read* cos 2 A.
10, *for* $\cos\frac{A}{2}$ *read* $\sqrt{2}\,.\cos\frac{A}{2}$.
„ 11 *for* $\text{cosec}^2 A$ *read* $\text{cosec}^2 2A$.
„ 12 *for* $-\sqrt{-1}$ *read* $-\sqrt{-1}\,.\sin A$.
77, Line 24, *for* $\frac{\sqrt{3}}{2}$ *read* $\frac{\sqrt{3}}{4}$.
93, „ 2, *for* $24r\frac{180}{24}$ *read* $24r\sin\frac{180}{24}$.
100, „ 3, *for* $ac(\theta+\phi)$ *read* $ac\sin(\theta+\phi)$.
102, Fig. 2, *for* C *read* D; and *for* D *read* C.
103, Line 5 from bottom, *for* $c\cos\beta$ *read* $c\cos\gamma$.

PREFACE.

In the compilation of this work, the most esteemed writers, both English and foreign, have been consulted, but those most used are De Fourcy and Legendre.

Napier's Circular Parts have been treated in a manner somewhat different to most modern writers. The terms conjunct and adjunct, used by Kelly and others, are here retained, as they appear to be more conformable to the practical views of Napier himself.

There are many other parts connected with Spherics that might be treated of, but which are not adapted to a Rudimentary Treatise like the present; those, however, who wish to see all the higher departments fully developed, must consult the writings of that distinguished mathematician, Professor Davies, of the Royal Military Academy, Woolwich.

Hutton's Course, the Ladies' and Gentleman's Diaries, (latterly comprised in one), Leybourne's Repository, the Mechanics' Magazine, and various other periodicals, teem with the productions of his fertile mind, both on this and other kindred subjects.

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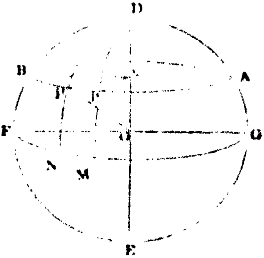
SPHERICS.

PRELIMINARY CHAPTER.

1. A SPHERE is a solid determined by a surface of which all the points are equally distant from an interior point, which is called the centre of the sphere.

2. Every section of a sphere made by a plane cutting it is the arc of a circle.



Let O be the centre of the sphere, A P B A a section made by a plane passing through it, draw O C to the cutting plane, and produce it both ways to D and E, and draw the radii of the sphere O A, O P.

Now, since O C P and O C A are right angles, $OA^2 - OC^2 = AC^2$, and $OP^2 - OC^2 = PC^2$, but $OA^2 = OP^2$; $\therefore AC^2 = PC^2$ or $AC = PC$; hence the section A P B A is a circle.

If the cutting plane pass through the centre, the radius of the section is evidently equal to the radius of the sphere, and such a section is called a great circle of the sphere.

3. The poles of any circle are the two extremities of that diameter or axis of the sphere which is perpendicular to the plane of that circle; and therefore either pole of any circle is equidistant from every part of its circumference, and, if it be a great circle, its pole is 90° from the circumference. A spherical triangle is the portion of space comprised between three arcs of intersecting great circles.

4. The angles of a spherical triangle are those on the surface of the sphere contained by the arcs of the great circles which form the sides, and are the same as the inclinations of the planes of those great circles to one another.

5. Any two sides of a spherical triangle are greater than the third side.

Since by Euclid XI. 20, any two of the plane angles, which form the solid angle at O, are together greater than the third, hence any two of the arcs which measure those angles must be greater than the third.

6. Since the solid angle at O (*see* fig. p. 3) is contained by three plane angles, and by Euclid XI. 21, these are together less than four right angles, hence the three arcs of the spherical triangle which measure those angles must be together less than the circumference of a great circle, that is $a + b + c > 360$, and since any two sides of a triangle is greater than the third, we have $a + b > c$; $b + c > a$; $a + c > b$.

ON THE POLAR OR SUPPLEMENTAL

7. If three arcs of great circles be described from the angular points A, B, C, of any spherical triangle A B C, as poles, the sides and angles of the new triangle, D F E, so formed will be the supplements of the opposite angles and sides of the other, and *vice versâ*.

Since B is the pole of D F, then B D is a quadrant, and since C is the pole of D E, C D is a quadrant; therefore the distances of the points B and C from D being each a quadrant, they are equal to each other, hence D is the pole of B C.

$$DE = 180 - C; \; EF = 180^\circ - A;$$

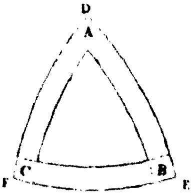
$$FD = 180^\circ - B; \text{ and } D = 180^\circ - BC;$$

$$E = 180^\circ - AC; \; F = 180^\circ - AB.$$

$$\text{Also, } AB = 180 - F \quad BC = 180^\circ - D;$$

$$AC = 180^\circ - E; \; A = 180 - FE;$$

$$B = 180^\circ - FD; \; C = 180^\circ - DE.$$



The sum of the three angles of a spherical triangle is greater than two right angles, and less than six right angles.

For if $a' + b' + c'$ be the sides of the supplemental or polar triangle, $A = 180^\circ - a'$; $B = 180^\circ - b'$; $C = 180^\circ - c'$;

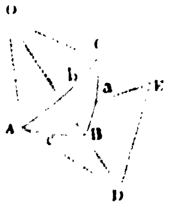
$$\text{hence } A + B + C + a' + b' + c' = 6 \times 90 = 6 \text{ right angles;}$$

but $a' + b' + c'$ is less than four right angles, by Euclid XI. 21; therefore $A + B + C$ is greater than two right angles; and as the sides a', b', c', of the polar triangle must have some magnitude, the sum of the three angles A, B, C must be less than six right angles.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

8. Spherical Trigonometry treats of the various relations between the sines, tangents, &c., of the known parts of a spherical triangle, and those that are unknown; or, which is the same thing, it gives the relations between the parts of a solid angle formed by the inclination of three planes which meet in a point, for the solid angle is composed of six parts, the inclinations of the three plane faces to each other, and also the inclinations of the three edges; in fact, a work might be written on this subject without using the spherical triangle at all, for the six parts of the spherical triangle are measures of the six parts of the solid angle at O. See fig.



9. If a spherical triangle have one of its angles a right angle, it is called a right-angled triangle; if one of its sides be a quadrant, it is called a quadrantal triangle; if two of the sides be equal, it is called an isosceles triangle, &c., as in Plane Trigonometry.

10. To determine the sines and cosines of a spherical triangle in terms of the sines and cosines of the sides.

Let O be the centre of the sphere on which the triangle ABC is situated, draw the radii OA, OB, OC; from OA draw the perpendiculars AD and AE, the one in the plane OAB, and the other in the plane OAC, and suppose them to meet the radii **O B** and **O C** produced in **D** and **E**. The angle **D A E** is equal to the angle **A** of the spherical triangle, and taking the radius unity we have $AD = \tan c$, $OD = \sec c$, $AE = \tan b$, $OE = \sec b$.

Then in triangles **D A E** and **D O E** we have

$$OD^2 + OE^2 - 2OD \, . \, OE \cos EOD = DE^2$$
$$AD^2 + AE^2 - 2AD \, . \, AE \cos A = DE^2$$

by subtracting the second equation from the first, observing that $OD^2 - AD^2 = OE^2 - AE^2 = 1$, and **E O D** is measured by **B C** or a, we obtain

$$2 + 2AD \, . \, AE \cos A - 2OD \, . \, OE \cos a = 0;$$

or by substituting the above values

$$1 + \tan b \, . \, \tan c \cos A - \sec b \sec c \cos a = 0$$

$$\text{but } \sec b = \frac{1}{\cos b}, \; \tan b = \frac{\sin b}{\cos b};$$

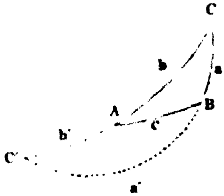
$$\sec c = \frac{1}{\cos c}, \; \tan c = \frac{\sin c}{\cos c};$$

$$\therefore \; 1 + \frac{\sin b \sin c \cos A}{\cos b \cos c} - \frac{\cos a}{\cos b \cos c} = 0;$$

hence $\cos a = \cos b \cos c + \sin b \sin c \cos A$ .... (1)

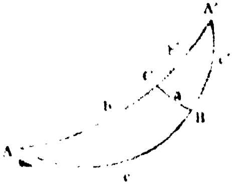
which is the fundamental formula in Spherical Trigonometry.

11. In the figure the sides b and c are less than 90°, but it is easily seen that equation (1) is general. Let us suppose that one of the sides, **A C** or b for example, is greater than 90°; draw the semi-circumferences **C A C′**, **C B C′**, and make the triangle **A B C′** of which the sides a' and b', or **B C′** and **A C′**, are supplements of a and b, and the angle **B A C′** the supplement of **A**. Since the sides b and c are less than 90°, the equation (1) can be applied to the triangle **A B C′**, and gives



$$\cos a' = \cos b' \cos c + \sin b' \sin c \cos BAC' \; ... \; (2).$$

Now $a' = 180° - a$, $b' = 180° - b$, $BAC' = 180° - A$; these values substituted in eq. (2) will give eq. (1), which shows that it is true for the case where b is greater than 90°.



Let us now suppose that the two sides b and c are both greater than 90°; produce AB and AC till they intersect in A', which forms the triangle BCA' in which the angle A' is equal to A, and the sides b' and c' the supplements of b and c; by making the substitutions in this case, we still find that equation (1) satisfied.

Lastly, we can verify equation (1) in the case where $b = 90°$ and $c = 90°$ either both together or separately.

If we apply equation (1) to each of the sides of the triangle, we shall have three equations by means of which we can always find any three parts whatever of the triangle, when the three others are given. But, for practice, it is necessary to have separately the divers relations which exist between four parts of the triangle taken in every possible manner. There are in all four distinct combinations, which we proceed to give.

12. 1st, Relation between the three sides and an angle.

By applying equation (1) to the three angles, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \;\ldots\ldots\; (1)$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \;\ldots\ldots\; (2)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \;\ldots\ldots\; (3)$$

13. 2nd, Relation between two sides and their opposite angles.

From equation (1) we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence $\sin^2 A = 1 - \cos^2 A = 1 - \dfrac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}$$

We must take the radical with the positive sign, seeing that the angles and the sides are less than 180°; their sines are positive. As the second member remains constant when we change A and a into B and b, &c., we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Hence in any spherical triangle, the sines of the angles are to each other as the sines of their opposite sides.

11. 3rd, Relation between the two sides and their included angle, and the angle opposite one of them.

In considering the combination a, b, A, C; first eliminate $\cos c$, between the equations (1) and (3) and we have

$$\cos a = \cos a \cos^2 b + \cos b \sin a \sin b \cos C + \sin b \sin c \cos A$$

transposing $\cos a \cos^2 b$, and observing that $\cos a - \cos a \cos^2 b = \cos a \sin^2 b$; and, dividing the whole by $\sin b \sin a$, it becomes

$$\frac{\cos a \sin b}{\sin a} = \cos b \cos C + \frac{\sin c \cos A}{\sin a},$$

but $\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}$; and consequently we have for the relation sought

$$\cot a \sin b = \cos b \cos C + \sin C \cot A.$$

By permuting the letters, we have in all the following six equations:

$$\cot a \sin b = \cos b \cos C + \sin C \cot A \ldots\ldots\ldots \quad (5)$$
$$\cot b \sin a = \cos a \cos C + \sin C \cot B \ldots\ldots\ldots \quad (6)$$
$$\cot a \sin c = \cos c \cos B + \sin B \cot A \ldots\ldots\ldots \quad (7)$$
$$\cot c \sin a = \cos a \cos B + \sin B \cot C \ldots\ldots\ldots \quad (8)$$
$$\cot b \sin c = \cos c \cos A + \sin A \cot B \ldots\ldots\ldots \quad (9)$$
$$\cot c \sin b = \cos b \cos A + \sin A \cot C \ldots\ldots\ldots \quad (10)$$

15. 4th, Relation between one of the sides and the three angles. Eliminate b and c from the equations (1) (2) (3): to do this we have by the last article

$$\frac{\cos a \sin b}{\sin a} = \cos b \cos C + \frac{\sin c \cos A}{\sin a},$$

$$\text{and since } \frac{\sin b}{\sin a} = \frac{\sin B}{\sin A} \text{ and } \frac{\sin c}{\sin a} = \frac{\sin C}{\sin A},$$

we have

$$\cos a \sin B = \cos b \sin A \cos C + \cos A \sin C;$$

and changing a and A into b and B, and *vice versâ*, we obtain

$$\cos b \sin A = \cos a \sin B \cos C + \cos B \sin C.$$

We have only now to eliminate $\cos b$ by the two preceding equations. We find after reduction the relation sought between A B C and a, which, applied to the three angles successively, will give the three equations

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \ \ldots\ldots\ (11)$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \ \ldots\ldots\ (12)$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \ \ldots\ldots\ (13)$$

16. The analogy of these equations with the fundamental formula is striking, and conducts us to a remarkable consequence. Let us imagine a spherical triangle A′ B′ C′, of which the sides a' b' c' are the supplements of the angles A, B, C; then from equation (1) we shall have

$$\cos a' = \cos b' \cos c' + \sin b' \sin b' \cos A'.$$

Now $\sin a' = \sin A$, $\cos a' = -\cos A$, $\sin b' = \sin B$, &c., then

$$-\cos A = \cos B \cos C + \sin B \sin C \cos A'.$$

From this equation we find for $\cos A'$ a value equal but of a contrary sign to that which we find for $\cos a$ in equation (11): then $a = 180^\circ - A'$, similarly $b = 180 - B'$, and $c = 180 - C'$. Hence, having given any spherical triangle, if we describe another triangle, the sides of which are the supplements of the angles of the first, then the sides of the first will be the supplements of the angles of the second. From this property the two triangles are called supplementary, and sometimes the triangles are said to be polar to each other.

NAPIER'S ANALOGIES.

17. We now proceed to deduce the formulæ known by the name of the analogies of Napier, which are employed to simplify some of the cases of spherical triangles.

The equations (1) and (2) give

$$\cos a - \cos b \cos c = \sin b \sin c \cos A\,;$$
$$\cos b - \cos a \cos c = \sin a \sin c \cos B.$$

By division, observing that $\dfrac{\sin a}{\sin b} = \dfrac{\sin A}{\sin B}$,

we have $\dfrac{\cos b - \cos a \cos c}{\cos a - \cos b \cos c} = \dfrac{\sin A \cos B}{\sin B \cos A}$.

By subtracting and adding unity to both sides of this equation and again dividing

$$\frac{\cos b - \cos a}{\cos b + \cos a} \times \frac{1 + \cos c}{1 - \cos c} = \frac{\sin (A - B)}{\sin (A + B)}.$$

But by Plane Trigonometry, page 30,

$$\frac{\cos b - \cos a}{\cos b + \cos a} = \tan \tfrac{1}{2} (a + b) \tan \tfrac{1}{2} (a - b)$$

but $\dfrac{1 + \cos c}{1 - \cos c} = \dfrac{1}{\tan^2 \frac{1}{2} c}$

and $\sin (A + B) = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A + B)$

$\sin (A - B) = 2 \sin \frac{1}{2} (A - B) \cos \frac{1}{2} (A - B)$.

Substituting these values, the above equation becomes

$$\tan \tfrac{1}{2} (a + b) \tan \tfrac{1}{2} (a - b) =$$
$$\tan^2 \tfrac{1}{2} c \left(\frac{\sin \frac{1}{2} (A - B) \cos \frac{1}{2} (A - B)}{\sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A + B)} \right) \cdot \quad (a)$$

and since $\dfrac{\sin a}{\sin b} = \dfrac{\sin A}{\sin B}$

we have $\dfrac{\sin a + \sin b}{\sin a - \sin b} = \dfrac{\sin A + \sin B}{\sin A - \sin B}$.

By Plane Trigonometry, page 30,

$$\frac{\tan \frac{1}{2} (a + b)}{\tan \frac{1}{2} (a - b)} = \frac{\sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)}.$$

Multiply these two equations together, and then dividing one by the other and extracting the root, observing that $\tan \frac{1}{2}(a+b)$ and $\cos \frac{1}{2}(A+B)$ ought to have the same sign,

$$\tan \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2} c \,.\, \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

$$\tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2} c \,.\, \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

To apply these to the polar triangle we must replace a, b, c, A, B, C, by $180° - A$, $180° - B$, $180° - C$, $180° - a$, $180° - b$, and there results,

$$\tan \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2} C \,.\, \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \quad \ldots\ldots\ldots\ldots (16)$$

$$\tan \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2} C \,.\, \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \quad \ldots\ldots\ldots\ldots (17)$$

My able and talented friend, Mr. Reynolds of Chelsea Hospital, has sent me the following very neat method of deducing Napier's Analogies, which he says was communicated to him by Mr. Adams, the celebrated astronomer of Cambridge

$$\text{Let } m = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin A + \sin B}{\sin a + \sin b}.$$

Then by the formula (11) page 7,

$$\begin{aligned} \cos A + \cos B \cos C &= \sin B \sin C \cos a \\ &= m \sin C \sin b \cos a \ldots (1) \\ \cos B + \cos A \cos C &= \sin A \sin C \cos b \\ &= m \sin C \sin a \cos b \ldots (2) \end{aligned}$$

Add (1) and (2), then

$$\{\cos A + \cos B\}(1 + \cos C) = m \sin C \sin (a+b).$$

$$\text{Also } \sin A + \sin B = m(\sin a + \sin b).$$

Dividing and reducing we have

$$\tan \frac{A+B}{2} . \tan \frac{C}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \quad \ldots\ldots\ldots\ldots\ldots (\text{Anal. } 1)$$

Again, subtracting (2) from (1)

$$(\cos B - \cos A)(1 - \cos C) = m \,.\, \sin(a - b) \sin C$$

$$\text{and } \sin B + \sin A = m(\sin a + \sin b);$$

∴ dividing

$$\tan \frac{A - B}{2} \,.\, \tan \frac{C}{2} = \frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}} \text{ .... (Anal. 2)}$$

The other two follow of course from the polar triangle.

ON RIGHT-ANGLED SPHERICAL TRIANGLES.

18. The preceding formulæ will apply to right-angled triangles, if we make any one of the angles = 90°.

If $A = 90°$ we have

$\cos a = \cos b \cos c$...... (1)
$\sin b = \sin a \sin B$...... (2) $\sin c = \sin a \sin C$...... (7)
$\tan b = \tan a \cos C$...... (3) $\tan c = \tan a \cos B$...... (8)
$\tan b = \sin c \tan B$...... (4) $\tan c = \sin b \tan C$...... (9)
$\cos B = \sin C \cos b$...... (5) $\cos C = \sin B \cos c$......(10)
$\cos a = \cot B \cot C$...... (6)

These six independent formulæ are all adapted to logarithmic calculation.

The first gives a relation between the hypothenuse and the two sides containing the right angle; the second, one side and angle opposite; the third, between the hypothenuse, a side, and the adjacent angle; the fourth, between the two sides and the angle opposite to one of them; the fifth, between one side and the two oblique angles; lastly, the sixth, between the hypothenuse and the oblique angles.

19. The formula (1) requires that $\cos a$ must have the same sign as the product $\cos b \cos c$, or that the three cosines must be positive, or that only one must be so. Therefore in any right-angled spherical triangle the three sides must be less than 90°; or two of them must be greater than 90°, and the third less. The formula (4) shows that $\tan b$ has the same sign as $\tan B$, and $\tan c$ the same sign as $\tan C$. Therefore each side containing the right angle is of the same kind or affection as the angle opposite, that is, the angle and the side are both less than 90° or both greater.

NAPIER'S CIRCULAR PARTS.

20. As we have before observed, the above formulæ are simple and well adapted for logarithmic computation, yet they are not easily remembered; therefore it is of importance that we should have some method which will relieve the memory as much as possible; this is supplied by what is termed Napier's Circular Parts. By committing to memory the two rules which will be given hereafter, the student will be able to solve all the cases in right-angled triangles, as well as if he had all the formulæ by heart.

The circular parts of a right-angled spherical triangle are five, namely, the two sides, the complement of the hypothenuse, and the complements of the two angles (the right angle being always omitted).

Three of these circular parts, besides the right angle, enter every proportion, two of which are given, and the third sought.

These three parts are named from their positions with respect to one another, that is, according as they are joined or disjoined, observing that the right angle does not separate the sides.

If the three circular parts join, that which is in the middle is called the *middle part*, and the other two are called *extremes conjunct*.

If the three circular parts do not join, two out of the five must, and that part which is separate or alone is the *middle part*, and the other two are called *extremes disjunct* *.

These things being understood, the following is the general rule.

The sine of the middle part is equal to the product of the tangents of the extremes conjunct.

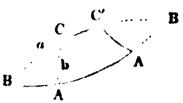
---

* Thus, if in figure page 12 we suppose B C, the angle B, and the side A B to be the quantities that are to be used; now as they lie all together, the angle B is the middle part, and the two sides, B C and A B, are the extremes conjunct. Also, if the angle B, A B and A C be the quantities, then since the right angle does not separate the sides, A B is the middle part, and the other two elements are the extremes conjunct. But if the quantities be A C, B C and the angle B, then the angle C is said to separate A C from B C, and the side A B is said to separate A C from the angle B, that part A C which is separated from both the others, call the middle part, and the parts which are disjoined from it call extremes disjunct. This practical method will be useful to seamen, and requires very little effort of memory.

The sine of the middle part, is equal to the product of the cosines of the extremes disjunct.

From these two equations, proportions may be formed, observing always to take the complements of the angles and hypothenuse; and that the cosine of a complement is a sine, and the tangent of a complement is a co-tangent, and *vice versâ.*

21. CASE 1. When the hypothenuse B C and the base A B are given to find the remaining parts of the triangle.



Let us first proceed to find A C.

Here the hypothenuse and the two sides are the three circular parts.

The hypothenuse being separated or disjoined from the sides it is the middle part, and the sides are the extremes disjunct.

Then sin B C = cos A B cos A C.

And since we must always take the complements of the hypothenuse and angles, this becomes

$$\cos a = \cos b \cos c.$$

Now, as this agrees with equation (1), the rule is proved in this case.

To find the angle B.

Here the three circular parts all lie together, taking B to be the middle part, then A B and B C are adjacent parts, or extremes conjunct.

$$\therefore \sin B = \tan BC \,.\, \tan AB;$$

taking the complements of B and B C, we have,

$$\cos B = \cot a \tan c,$$

which corresponds with (8), and therefore the rule is proved in this case also.

To find the angle C.

Here the side A B is separated from the hypothenuse by the angle B, and it is separated from the angle C by the side A C, then A B being the middle part, the hypothenuse and the required angle are the extreme or disjoined parts.

$$\sin AB = \cos BC \cos C;$$

taking the complements of B C and C,

$$\sin c = \sin a \sin C.$$

This agrees with (7), and therefore proves the rule.

22. Case 2. Given the two sides b, and c, which include the right angle, to find the hypothenuse and the angles.

1. To find the hypothenuse.

As the two sides are separated from the hypothenuse they will be extremes disjoined, the hypothenuse being the middle part;

$$\sin BC = \cos AB \cos AC;$$

taking the complement of BC,

$$\cos a = \cos b \cos c; \text{ which is the same as equation (1)};$$

$$\therefore \cos c = \frac{\cos a}{\cos b}.$$

To find angle C.

Since the right angle A does not disjoin, the three parts all lie together, hence AC being the middle part, AB and angle C are the adjacent parts, or extremes conjunct.

$$\sin AC = \tan AB \,.\, \tan C;$$

taking the complement of C,

$$\sin b = \tan c \cot c; \text{ which agrees with (9)};$$

$$\text{or } \tan C = \frac{\tan c}{\sin b}.$$

To find the angle B.

The three circular parts all lie together again, AB being in the middle; calling it the middle part, then AC and angle B will be extremes conjunct.

$$\sin AB = \tan AC \tan B;$$

taking the complement of B,

$$\sin c = \tan b \cot B, \text{ which agrees with (4)};$$

$$\text{or } \tan B = \frac{\tan b}{\sin c}.$$

23. Case 3. Given the hypothenuse a and angle B to find b, c, C.

1. To find AC or b.

As AC is separated from the hypothenuse by the angle C, and from the angle B by the side AB; calling AC the middle part, then BC and angle B are extremes disjunct.

$$\sin AC = \cos BC \cos B;$$

taking the complements of BC and C,

$$\sin b = \sin a \sin B, \text{ which is the same as equation (2)}.$$

2. To find AB or c.

Here the three circular parts all lie together, and AB in the middle; calling it the middle part, then AB and BC will be adjacent parts, or extremes conjunct.

$$\sin B = \tan AB \tan BC;$$

taking the complements of B and BC,

$$\cos B = \tan c \cot a;$$

$$\therefore \tan c = \frac{\cos B}{\cot a} = \cos B \tan a, \text{ which agrees with equ. (8).}$$

3. To find C.

Here the circular parts lie all together, the hypothenuse being in the middle; call it the middle part, and the angles B and C will be adjacent parts.

$$\sin BC = \tan B \tan C;$$

taking the complements throughout,

$$\cos a = \cot B \cot C;$$

$$\therefore \cot C = \frac{\cos a}{\cot B} = \cos a \tan B, \text{ which agrees with equ. (6).}$$

24. CASE 4. Given the side AC or b and the opposite angle B to find a, c, C.

1. To find the hypothenuse BC or a.

Here b or AC is separated from the hypothenuse by the angle C, and from the angle B by the side AB; calling then AC the middle part, the angle B and the hypothenuse are the extremes disjunct.

$$\sin AC = \cos BC \cos B;$$

taking the complements of BC and B,

$$\sin b = \sin a \sin B \text{ which agrees with equation (2).}$$

$$\sin a = \frac{\sin b}{\sin B}.$$

2. To find c.

As the right angle does not disjoin, c lies in the middle between b and B; calling it the middle part, b and B are the extremes conjunct.

$$\sin C = \tan b \tan B;$$

taking the complement of B,

$$\sin c = \tan b \cot B, \text{ which agrees with equation (4).}$$

3. To find C.

Here the angle B is separated from c by the hypothenuse, and it is separated from b by the side AB; calling it the middle part,

$$\sin B = \cos b \cos C;$$

taking the complements of B and C,

$$\cos B = \cos b \sin C, \text{ which agrees with equation (5).}$$

$$\sin C = \frac{\cos B}{\cos b}.$$

There is here an ambiguity, since each quantity is determined by its sine, and we see that this really ought to be the case. In fact, if the triangle BAC (fig. p. 12) right-angled at A, satisfy the equation; produce BA and BC till they intersect in D, then take DA′ = BA, and DC′ = BC, the triangles BAC, DA′C′ will be equal in all respects, then the angle A is a right angle, and C′A′ = CA = b. Thus the triangle BA′C′ is right angled, and contains also the given parts B and b; we can therefore take at will $a < 90°$, or $A > 90°$, but when the choice is once made the affection or species of c will be determined by the equation $\cos a = \cos b \cos c$, and that affection will be the same as that of C. There will be only one triangle which has two right angles when $b = B$, and none when we have $\sin b > \sin b$.

25. Case 5. Given the side b and the adjacent angle C, to find a, c, B.

1. To find the hypothenuse a.

Here the parts all lie together, the angle C being in the middle; call it the middle part, then AC and BC are the extremes conjunct.

$$\sin C = \tan AC \tan BC;$$

taking the complements of C and BC,

$$\cos C = \tan b \cot a; \text{ which agrees with equation (3);}$$

$$\therefore \tan a = \frac{\tan b}{\cos C}.$$

2. To find AB or c.

As the right angle does not disconnect, AC is the middle part, and AB and angle C are the extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C,

$$\sin b = \tan c \cot c\,;$$

$$\therefore \ \tan c = \frac{\sin b}{\cot \mathrm{C}} = \sin b \tan \mathrm{C}, \text{ which agrees with equ. (9).}$$

3. To find the angle B.

Since the angle B is separated from b by the side AB, and from the angle C by the hypothenuse BC; calling it the middle part, then AC and the angle C are the extremes disjunct.

$$\sin \mathrm{B} = \cos \mathrm{AC} \cos \mathrm{C}\,;$$

taking the complements of B and C,

$$\cos \mathrm{B} = \cos b \sin \mathrm{C}, \text{ which agrees with equation (5).}$$

Here a, c and B are found without any ambiguity.

26. CASE 6. Given the two oblique angles B and C to find a, b, c.

1. To find a or BC.

Here a, B and C all lie together, a or BC being in the middle; call it the middle part, then B and C are the extremes conjunct.

$$\sin \mathrm{BC} = \tan \mathrm{B} \tan \mathrm{C}\,;$$

taking the complements of the whole,

$$\cos a = \cot \mathrm{B} \cot \mathrm{C}, \text{ which agrees with equation (6).}$$

2. To find b or AC.

Here B is separated from C by the hypothenuse BC, and it is separated from AC by the side AB; calling it the middle part, then the angle C and AC are the extremes disjunct.

$$\sin \mathrm{B} = \cos \mathrm{C} \cos \mathrm{AC}\,;$$

taking the complements of B and C,

$$\cos \mathrm{B} = \sin \mathrm{C} \cos b, \text{ which agrees with equation (5).}$$

3. To find c or AB.

Here the angle C is separated from AB by the side AC, and from the angle B by the hypothenuse BC; calling it the middle part, then AB and the angle B are the extremes disjunct.

$$\sin \mathrm{C} = \cos \mathrm{AB} \cos \mathrm{B}\,;$$

taking the complements of C and B,

$$\cos \mathrm{C} = \cos c \sin \mathrm{B}, \text{ which agrees with equation (10);}$$

$$\therefore \cos c = \frac{\cos C}{\sin B}.$$

These values leave no ambiguity, and if the triangle is impossible they will show that it is so.

27. When a triangle is isosceles, the two equal sides are only counted as one element, and the angles which are opposite to them also as only one element. Now, if we draw the arc of a great circle through the vertex of the triangle and the middle of the base, we divide it into two right-angled triangles, equal in all respects, and in each of which we know two elements besides the right angle, then the isosceles triangle can be solved by the formulæ for right-angled triangles.

28. If in a spherical triangle ABC, in which we have $a + b = 180^\circ$, produce a and c till they intersect in D, we shall have $a + \mathrm{CD} = 180^\circ$, hence $\mathrm{CD} = b$; therefore, the solution of the triangle ABC is brought to that of the isosceles triangle ABC.

The same thing may be said of a triangle, in which two angles are the spherical supplements of each other, for we cannot have $\mathrm{A} + \mathrm{B} = 180$ without at the same time having $\mathrm{A} + \mathrm{B} = 180^\circ$ and *vice versâ*. In fact, in the isosceles triangle ACD, the angle $\mathrm{CAD} = \mathrm{D} = \mathrm{B}$. Now, $\mathrm{CAD} + \mathrm{CAB} = 180^\circ$; then also, in the triangle ABC we ought to have $\mathrm{A} + \mathrm{B} = 180^\circ$.

CHAPTER II.

SOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

29. CASE 1. Given the three sides a, b, c to find the angles A, B, C.

To find A, we have by equation (1)

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

but we obtain an expression better adapted to logarithms by finding $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, &c., as in Plane Trigonometry.

Since $2 \sin^2 \frac{1}{2} A = 1 - \cos A$, we have by substituting the above value of $\cos A$,

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} =$$

(by equation (8) page 30, Plane Trigonometry.)

$$\frac{2 \sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c};$$

$$\therefore \sin \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c}}.$$

For the sake of abridgment, put $a + b + c = 2s$, and the preceding expression becomes

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}$$

In the same way

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}};$$

$$\therefore \tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

30. CASE 2, Given the two sides a, b, and the angle A opposite to one of them, to find c, B, C.

We obtain at first the angle B opposite to b by the proportion

$$\sin a : \sin b :: \sin A : \sin B;$$

$$\therefore \sin B = \frac{\sin A \sin b}{\sin a}.$$

It will be best to determine c and C by Napier's Analogies, which give

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a - b) \cdot \frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} (A - B)}$$

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A - B) \cdot \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)}$$

As the angle B is determined by its sine, it can either be acute or obtuse. However, for certain values of the given quantities a, b, A, there will be only one triangle. We may refer back to the similar case of plane triangles, we can thus find C in a direct manner by the equation

$$\cot A \sin C + \cos b \cos C = \cot a \sin b.$$

To effect this, let us at first determine an auxiliary angle φ, by putting $\cot A = \cos b \cot \varphi$, from whence we have

$$\cot \varphi = \frac{\cot A}{\cos b};$$

then in the equation (5), p. 6, $\cot A = \cos b \cot \varphi = \dfrac{\cos b \cos \varphi}{\sin \varphi}$, the equation becomes

$$\cos b (\sin C \cos \varphi + \cos C \sin \varphi) = \cot a \sin b \sin \varphi,$$

from which we find

$$\sin (C + \varphi) = \frac{\tan b \sin \varphi}{\tan a};$$

hence $C + \varphi$ is determined; let $C + \varphi = m$, and we have $C = m - \varphi$.

After having found C, we obtain the side c by the proportion

$$\sin A : \sin C :: \sin a : \sin c.$$

But if we wish to find c directly, we must refer back to equation (1), page 5,

$$\cos b \cos c + \cos A \sin b \sin c = \cos a.$$

This may be reduced in the same way as the equation above, by using an auxiliary angle ϕ, putting $\cos A \sin b = \cos b \cot \phi$, from whence we have

$$\cot \phi = \cos A \tan b;$$

consequently, the above equation becomes

$$\cos b (\sin \phi \cos c + \cos \phi \sin c) = \cos a \sin \phi, \text{ or}$$

$$\sin (c + \phi) = \frac{\cos a \sin \phi}{\cos b}.$$

Having found ϕ, we can easily find c.

31. CASE 3. Given the two sides a and b and the included angle C to find A, B, c.

The formulæ (5) (6), page 6, give for A and B

$$\cot A = \frac{\cot a \sin b - \cos b \cos C}{\sin C}$$

$$\cot B = \frac{\cot b \sin a - \cos a \cos C}{\sin C}$$

By employing auxiliary angles it is easy to reduce each numerator to a single quantity, but is is more simple to recur to Napier's Analogies.

$$\tan \tfrac{1}{2} (A + B) = \cot \tfrac{1}{2} C \,.\, \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)}$$

$$\tan \tfrac{1}{2} (A - B) = \cot \tfrac{1}{2} C \,.\, \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)}$$

which give $\frac{1}{2} (A + B)$ and $\frac{1}{2} (A - B)$, consequently by adding and subtracting we find A and B.

The angles being found we obtain c from the proportion $\sin A : \sin C :: \sin a : \sin c$; but if we wish to have c directly we must take the formula, page 5.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C;$$

in which if we make $\sin b \cos C = \dfrac{\cos b \cos \phi}{\sin \phi}$

$= \cos b \cot \phi$, then it becomes without any ambiguity

$$\cot \phi = \tan b \cos C, \therefore \cos c = \frac{\cos b \sin (a + \phi)}{\sin \phi}.$$

32. CASE 4. Given the two angles A and B, and the adjacent side c, to find a, b, c.

We can find a and b by the formulæ (7) and (9), page 6,

$$\cot a = \frac{\cot A \sin B + \cos B \cos c}{\sin c}$$

$$\cot b = \frac{\cos B \sin A + \cos A \cos c}{\sin c}$$

and better still by Napier's Analogies,

$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c \,.\, \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}$$

$$\tan \tfrac{1}{2}(a - b) = \tan \tfrac{1}{2} c \,.\, \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}.$$

These equations determine $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$, and from which, by adding and subtracting, we find a and b.

We can now find C by the proportion

$$\sin a : \sin c :: \sin A : \sin C,$$

or we can find C directly by making use of the formula, equation (13), page 7, viz.,

$$\cos C = \sin A \sin B \cos c - \cos A \cos B.$$

If we put $\sin B \cos c = \cos B \cot \varphi$, it will become

$$\cot \varphi = \tan B \cos c, \cos C = \frac{\cos B \sin (A - \varphi)}{\sin \varphi}.$$

This case is analogous to the third case, and offers no ambiguity.

33. Case 5. Given the two angles A and B, and the side a opposite to one of them, to find b, c, C.

This case is quite analogous to the second, and is treated in the same manner, and has the same ambiguities.

We deduce b from the proportion

$$\sin A : \sin B :: \sin a : \sin b,$$

and we find c and C by the formulæ already employed,

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2}(a - b) \,.\, \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)},$$

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A - B) \,.\, \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)}.$$

The side c can also be obtained by equation (7),

$$\cot a \sin c - \cos B \cos c = \cot A \sin B,$$

in which we make $\cot a = \cos B \cot \varphi$

$$\therefore \cot \phi = \frac{\cot a}{\cos B}, \quad \sin (c - \phi) = \frac{\tan B \sin \phi}{\tan A}.$$

Lastly, we can find C, for $\sin a : \sin c :: \sin A : \sin C$, or better by means of the equation,

$$\cos a \sin B \sin C - \cos B \cos C = \cos A,$$

we reduce the first member to a monomial by putting
$\cos a \sin B = \cos B \cot \phi$, from whence we have

$$\cot \phi = \cos a \tan B, \quad \sin (C - \phi) = \frac{\cos A \sin \phi}{\cos B};$$

these values determine ϕ, $C - \phi$, and consequently the angle C.

34. CASE 6. Given the three angles A, B, C, to find the sides a, b, c.

This case is solved in a similar way to the first.

By page 7, equation (11)

$$\cos a = \frac{\cos a + \cos B \cos C}{\sin B \sin C},$$

and by the same method, as used in the first case,

$$\sin \tfrac{1}{2} = a \sqrt{\frac{\sin S \sin (A - S)}{\sin B \sin C}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\sin (B - S) \,.\, \sin (C - S)}{\sin B \sin C}}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{\sin S \sin (A - S)}{\sin (B - S) \sin (C - S)}}.$$

By using the polar triangle in Case 1, we have

$$\sin \tfrac{1}{2} a = \sqrt{\frac{- \cos S \,.\, \cos (S - A)}{\sin B \sin C}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}$$

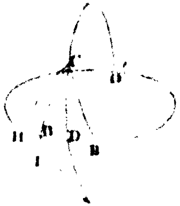
$$\tan \tfrac{1}{2} a = \sqrt{\frac{- \cos S \ \cos (S - A)}{\cos (S - B) \cos (S - C)}}.$$

The first and last of these appear under an impossible form, but since S is always greater than 90 and less than 270, the cos S is always negative, and therefore makes the quantity under the radical always positive.

ON THE AMBIGUOUS CASES OF SPHERICAL TRIANGLES.

35. The only cases in which there is any uncertainty are the second and fifth. We proceed to show in this article what conditions are necessary that there may be two solutions, or only one, or even when the triangle is impossible.

Let us consider upon a sphere a semicircle DCD′ perpendicular to a whole circle DHD′; take CD less than 90°, and draw the arcs of great circles CB, CB′, CH . . . from the point C to the different points of the circumference DHD′. Produce CD, making C′D = CD, and join C′B. The triangles CDB, C′DB have a right angle contained between the equal sides, therefore CB = C′B. Now we have $CDC' < CB + BC'$, therefore $CD < CB$.



Hence, in the first place, the arc CD is the least that we can draw from the point C to the circumference DHD′; and consequently CD′ is the greatest.

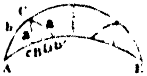
Let DB′ = DB; then in the two triangles CDB and CDB′ have the two sides CD, CB and the right angle CDB of the one, equal to the two sides CD, DB, and the right angle CDB′ of the other, hence CB′ = CB. Therefore, in the second place, the oblique arcs equally distant from CD or CD′ are equal.

Lastly, let $DH > DB$; draw C′H and produce CB till it intersects C′H in I. Then, since the arc CC′ is less than a semicircle, it will meet CB produced beyond the point C′; this requires that the intersection I falls between H and C′. We have therefore $C'B < C'I + IB$, and consequently $C'B + BC < C'I + IC$. But we have $IC < IH + HC$, and therefore $C'I + IC < C'H + HC$; hence, *à fortiori*, $C'B + BC < C'H + HC$. Now, $C'B = BC$ and $C'H = HC$, therefore we have $BC < HC$. Consequently, in the third place, the oblique arcs are greater the farther they are from CD, or the more they approach CD′.

Now, suppose we have to construct a spherical triangle, the given quantities being a, b, and the angle A opposite to a.

We may at first remark that certain cases of impossibility are indicated even by the calculation. To show this, make the angle CAB = A and AC = b, produce AC and AB till they intersect in E, then let fall the perpendicular CD upon AE.

The arc CD ought to be of the same affection or species as the angle A by Art. 19; then, when A is acute, CD is the shortest distance from the point C to the semi-circumference AE, and it is the greatest when A is obtuse.



In the first hypothesis the triangle will be impossible if we have $a <$ CD, which gives $\sin a < \sin$ CD; and in the second it will be impossible if we have $a >$ CD, which gives again $\sin$ A $> \sin$ CD.

Now, in the right-angled spherical triangle ACD, we have

$$\sin \mathrm{CD} = \sin b \sin \mathrm{A};$$

then, in both hypotheses we shall have $\sin a < \sin b \sin$ A. On the other hand, when we seek the angle B of the unknown triangle ACB, we have

$$\sin \mathrm{B} = \frac{\sin b \sin \mathrm{A}}{\sin a};$$

then this value of sin B will be > 1, which is impossible.

If we have $a =$ CD, there will be only one right-angled triangle, ACD, which will be possible, and it is that which again indicates the value of sin B, which becomes sin B = 1. It is understood that the angle A is not equal to 90°.

Let us now examine the different relations of magnitude which the given quantities a, b, A can present.

Let A $< 90°$ and $b < 90°$ (fig. p. 23). Since A and b are $< 90°$, AD is also $< 90°$ by Art. 19; then AD $<$ DE; if now we have besides $a < b$, it is clear that we can place between CA and CD an arc CB $= a$, and that on the other side, between CD and CE, we can put another CB' = CB $= a$: that is to say, there are two triangles ACB and ACB' which have the same quantities given, viz., a b A.

When $a = b$, the triangle ACB disappears, and there remains only the triangle ACB'.

When $a + b = 180$, or when $a + b > 180$, the point B' coincides with E, or passes beyond it, and then no triangle can exist.

We can discuss in the same manner the other hypotheses. The results are all contained in the following table. The sign $\ngtr$ signifies equal to or greater than; and the sign $\nless$ signifies equal to or less than.

$A < 90°$	$b < 90°$	$a < b$	two solutions.
		$a \ngtr b$	one solution.
		$a + b \ngtr 180°$	no solution.
	$b > 90°$	$a + b < 180°$	two solutions.
		$a + b \ngtr 180°$	one solution.
		$a \ngtr b$	no solution.
	$b = 90°$	$a < b$	two solutions.
		$a \ngtr b$	no solution.
$A > 90°$	$b < 90°$	$a + b > 180°$	two solutions.
		$a + b \nless 180°$	one solution.
		$a \nless b$	no solution.
	$b > 90°$	$a > b$	two solutions.
		$a \nless b$	one solution.
		$a + b \nless 180°$	no solution.
	$b = 90°$	$a > b$	two solutions.
		$a \nless b$	no solution.
$A = 90°$	$b < 90°$	$a > b$	one solution.
		$a \nless b$	no solution.
		$a + b \ngtr b$	no solution.
	$b > 90°$	$a < b$	one solution.
		$a \ngtr b$	no solution.
		$a + b \nless 180°$	no solution.
	$b = 90°$	$a = 90°$	solutions ad infinitum.
		$a < \text{or} > 90°$	no solution.

By the properties of the polar triangle, we can apply the results to the fifth case, where A, B, a, are given, only taking care to change a, b, A into A, B, a, the sign $>$ into $<$, and the sign $<$ into $>$.

When the given quantities fall in a case where we ought to have only one solution, the calculation will still indicate two. But to discern which ought to be taken, it is sufficient to observe, that the greater angle must be opposite to the greater side, and conversely.

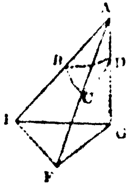
See *Lefebure De Fourcy's Trigonometry.*

Suppose, for example, that we have given $A = 112°$, $a = 102°$, $b = 106°$. In the preceding table, among the cases which correspond to $A > 90°$, we consider that where $b > 90°$, and among these that where $a \nless b$. We may observe besides, that $a + b = 208°$, therefore $a + b > 180°$, we conclude from the table that there is only one solution, and since b is $> a$, the angle B is greater than A, therefore B is obtuse.

TO REDUCE AN ANGLE TO THE HORIZON.

36. Let BAC be an angle in an inclined plane, and AD the vertical passing through A. Draw the horizontal plane meeting the lines AB, AC, AD, in E, F, G; the angle EGF is the horizontal projection of the angle BAC, or, in other words, it is the angle BAC reduced to the horizon. It is this angle EFG that we have to calculate, supposing the angles BAC, BAD, CAD, to have been determined by an instrument.

The geometrical construction is easy, for the line AG being arbitrary, we shall have sufficient quantities given to construct at first the right-angled triangle, EAG and FAG, then the triangle EAF, and, lastly, the triangle EGF. The calculation of the angle EGF is equally easy. If we describe a sphere from the centre A with any radius, the lines AB, AC, AD, where they meet the sphere, will determine a spherical triangle BCD, of which the sides are known by means of the given angles, and of which the angle BDC of the triangle is equal to the required angle EGF.



Then by the first case of oblique-angled spherical triangles, page 18, we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}};$$

where $a = BAC$; $b = BAD$; $c = CAD$; $s = \frac{1}{2}(a + b + c)$.

Let $a = 47° 45′ 39″$, $b = 69° 49′ 19″$, $c = 80° 17′ 36″$. We shall have $2s = 197° 52′ 34″$, $s = 98° 56′ 17″$; $s - b = 29° 6′ 58″$; $s - c = 18° 38′ 41″$.

$\log \sin (s-b)$	....	9·6871552
$\log \sin (s-c)$	....	9·5047412
comp. $\log \sin b$	....	0·0275078
comp. $\log \sin c$	....	0·0062623
$2 \log \sin \frac{1}{2} A$		19·2256665
$\log \sin \frac{1}{2} A$		9·6128332

$$\therefore \tfrac{1}{2} A = 24^\circ\ 12'\ 27''\cdot 9, \text{ or } A = 48^\circ\ 24'\ 56''.$$

37. The following properties of spherical triangles we shall premise before entering on the numerical solution of triangles.

Any side of a spherical triangle is less than a semicircle, and any angle is less than two right angles.

For the limit of any plane angle is two right angles, and this is also the limit of any plane face of a solid angle.

The sum of the three angles is greater than two right angles and less than six right angles.

If the three sides of a spherical triangle be equal, the three angles will also be equal, and *vice versâ*.

If the sum of any two sides of a spherical triangle be equal to 180°, the sum of their opposite angles will also be equal to 180°, and *vice versâ*.

If the three angles of a spherical triangle be all acute, all right, or all obtuse, the three sides will be accordingly all less than 90°, all equal to 90°, or all greater than 90°, and *vice versâ*.

The sum of any two sides is greater than the third side, and their difference is less than the third side.

The sum of any two angles is greater than the supplement of the third angle.

The sum of the three sides is less than the circumference of a great circle.

If any two sides of a triangle be equal to each other, their opposite angles will be equal, and *vice versâ*.

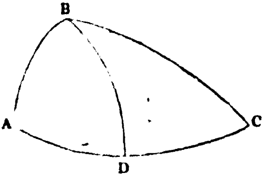
Since

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}.$$

If $b = a$, these expressions are each $= \dfrac{\cos a - \cos a \cos c}{\sin a \sin c}$

$$\therefore \cos A = \cos B \text{ or } A = B;$$



that is, the angles at the base of an isosceles triangle are equal, and the converse of this holds also. From this it is easily shown that the greater side of a spherical triangle is opposite the greater angle, for let A B C be greater than C A B, and make the angle A B D equal to the angle D A B; ∴ D A = D B.

$$A C = A D + D C = D C + D B, \text{ but } D C + D B > B C$$

$$\therefore A C > B C.$$

ON THE NUMERICAL SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

38. When the hypothenuse and one side are given.

Ex. 1. Given the hypothenuse B C = 63° 56′ 7″, and the side A B = 40°, to find the remaining parts of the triangle.

To find the other side, A C.

Here the hypothenuse and the two sides are the three circular parts.

The hypothenuse being separated or disjoined from the sides by the angles is therefore the middle part, and the sides the extremes disjunct.

$$\sin B C = \cos A B \cos A C;$$

taking the complement of hypothenuse as directed by the rule,

$$\cos B C = \cos A B \cos A C;$$

$$\log \cos B C = \log \cos A B + \log \cos A C - 10$$

$$\log \cos A C = \log \cos B C - \log \cos A B + 10$$

$$= \log \cos 63° 56' 7'' - \log \cos 40° + 10$$

$$= 9{\cdot}6428464 - 9{\cdot}8842540 + 10$$

$$= 9{\cdot}7585924;$$

$$\therefore A C = 54° 59' 59''{\cdot}6.$$

The side A C is acute, because the hypothenuse and the given side have the same affection.

To find the angle B.

This angle connects the hypothenuse and the given side, and is therefore the middle part, and the other the extremes conjunct.

$$\therefore \sin B = \tan AB \,.\, \tan BC;$$

taking the complements of the angle and hypothenuse,

$$\cos B = \tan AB \,.\, \cot BC;$$

$$\begin{aligned}\log \cos B &= \log \tan AB + \log \cot BC - 10\\ &= \log \tan 40^\circ + \log \cot 63^\circ\, 56'\, 7''\\ &= 9{\cdot}9238135 + 9{\cdot}6894258 - 10\\ &= 9{\cdot}6132393;\end{aligned}$$

$$\therefore\ B = 65^\circ\, 46'\, 5''.$$

The angle B is acute, as the hypothenuse and given side are of the same affection.

To find the angle C.

Here the side A B is separated from the hypothenuse by the angle B, and it is separated from the angle C by the side A C; take it to be the middle part, then BC and the angle A are extremes disjunct.

$$\sin AB = \cos BC \,.\, \cos C;$$

taking the complements of hypothenuse and angle C,

$$\sin AB = \sin BC \sin C;$$

$$\log \sin AB = \log \sin BC + \log \sin C - 10,$$

$$\begin{aligned}\log \sin C &= \log \sin AB - \log \sin BC + 10\\ &= \log \sin 40^\circ - \log \sin 63^\circ\, 56'\, 7'' + 10\\ &= 9{\cdot}8080675 + 0{\cdot}0465794 \text{ by taking comp. log. } 63^\circ\, 56'\, 7'';\\ &= 9{\cdot}8546469;\end{aligned}$$

$$\therefore\ C = 45^\circ\, 41'\, 21''.$$

The angle C is acute, the hypothenuse and given side being of the same affection.

When the two sides are given.

Given the side A C $= 52° 13'$, and the side **A B** $= 42° 17'$, to find the remaining parts.

To find the angle B. (See fig. p. 12.)

As the right angle does not disjoin, A B is the middle part, and the angle and side A C are extremes conjunct.

$$\sin AB = \tan B \cdot \tan AC;$$

taking the complement of B,

$$\sin AB = \cot B \tan AC;$$

$$\begin{aligned}\log \cot B &= \log \sin AB - \log \tan AC + 10\\ &= 9{\cdot}8278843 + 10 - 10{\cdot}1105786\\ &= 9{\cdot}7173057, \text{ which is the log cot } 62° 27';\end{aligned}$$

$$\therefore \; B = 62° 27',$$

which is acute, like its opposite side.

To find the angle C.

Here AC is the middle part, and the angle C and AB are extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C

$$\sin AC = \tan AB \cot C;$$

$$\begin{aligned}\log \cot C &= \log \sin AC - \log \tan AB + 10\\ &= 9{\cdot}8978103 - 9{\cdot}9587542 + 10\\ &= 9{\cdot}9390561, \text{ which is the log cot of } 49°.\end{aligned}$$

The angle is acute like its opposite side.

To find the hypothenuse B C.

The hypothenuse being separated from the sides by the angles, it is the middle part, and the sides are the extremes disjunct.

$$\sin BC = \cos AB \cdot \cos AC;$$

taking the complement of the hypothenuse,

$$\cos BC = \cos AB \cos AC;$$

$$\log\cos BC = \log\cos AB + \log\cos AC - 10$$
$$= 9{\cdot}8691301 + 9{\cdot}7872317 - 10$$
$$= 9{\cdot}6563613, \text{ which is the cosine } 62^\circ\ 31',$$

which is less than 90°, AC and BC being alike.

When a side and its opposite angle are given.

Given the side AC = 55°, and its opposite angle B = 65° 46′ 5″, to find the remaining parts of the triangle.

To find the other angle C.

Here B is the middle part, being separated from AC by AB, and from the angle C by BC;

∴ AC and C are the extremes disjunct.

$$\sin B = \cos AC \cos C;$$

taking the complements of B and C,

$$\cos B = \cos AC \sin C;$$

$$\log\sin C = \log\cos B - \log\cos AC + 10$$
$$= 9{\cdot}6132407 + \text{comp. }\log 0{\cdot}2411087 + 10$$
$$= 9{\cdot}8546494 = \log\sin 45^\circ\ 41'\ 21'';$$
$$\therefore\ C = 45^\circ\ 41'\ 21''.$$

The angle C is ambiguous; as it cannot be determined by the data alone whether, AB, C, and BC are greater or less than 90°.

To find the side AB.

Here AB is the middle part, AC and B the extremes conjunct.

$$\sin AB = \tan AC \tan B;$$

taking the complement of B,

$$\sin AB = \tan AC \cot B;$$

$$\log\sin AB = \log\tan AC + \log\cot B - 10$$
$$= 10{\cdot}1547732 + 9{\cdot}6532976 - 10$$
$$= 9{\cdot}8080708, \text{ which is the } \sin 40^\circ;$$
$$\therefore\ AB = 40^\circ.$$

The side AB is also ambiguous for the same reason as above.

To find the hypothenuse BC.

The side AC is the middle part, and BC and B are the extremes disjunct.

$$\sin AC = \cos BC \,.\, \cos B;$$

taking the complements of hypothenuse and angle, B,

$$\sin AC = \sin BC \sin B;$$

$$\log \sin AC = \log \sin BC + \log \sin B - 10$$

$$\log \sin BC = \log \sin AC - \log \sin B + 10$$

$$= 9{\cdot}9133645 + 0{\cdot}0400568 + 10$$

$$= 9{\cdot}9534213;$$

$$\therefore\ BC = 63^\circ\ 56'\ 7''.$$

When a side and its adjacent angle are given.

Given the side AC = 54′ 40′, and its adjacent angle 47° 56′, to find the remaining parts.

To find the side AB.

Here the circular parts all lie together, hence AC is the middle part, and AB and C the extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C.

$$\sin AC = \tan AB \cot C;$$

$$\log AC = \log \tan AB + \log \cot C - 10$$

$$\log \tan AB = \log \sin AC - \log \cot C + 10$$

$$= 9{\cdot}9121207 - 9{\cdot}9554535 + 10$$

$= 9{\cdot}9566672$ which is the tangent of 42° 8′ 46″;

$$\therefore\ AB = 42^\circ\ 8'\ 46''$$

which is acute, like its opposite angle.

To find the angle B.

Here B is separated from the two given quantities; calling it the middle part, then AC and C are the extremes disjunct.

$$\sin \mathrm{B} = \cos \mathrm{AC} \cos \mathrm{C};$$

taking the complements of B and C,

$$\cos \mathrm{B} = \cos \mathrm{AC} \sin \mathrm{C};$$

$$\begin{aligned} \log \cos \mathrm{B} &= \log \cos \mathrm{AC} + \log \sin \sin \mathrm{C} - 10 \\ &= 9{\cdot}7611063 + 9{\cdot}8706179 - 10 \\ &= 9{\cdot}6317242, \text{ which is } \cos 64^\circ\, 38'\, 31''; \\ \therefore\ \mathrm{B} &= 64^\circ\, 38'\, 31''. \end{aligned}$$

To find the hypothenuse BC.

Here the circular parts all lie together, and C being in the middle, is the middle part, and BC and AC the extremes disjunct.

$$\sin \mathrm{C} = \tan \mathrm{BC} \tan \mathrm{AC};$$

taking the complements of the hypothenuse and of angle C,

$$\cos \mathrm{C} = \tan \mathrm{AC} \cot \mathrm{BC};$$

$$\log \cos \mathrm{C} = \log \tan \mathrm{AC} + \log \cot \mathrm{BC} - 10;$$

$$\begin{aligned} \therefore\ \log \cot \mathrm{BC} &= \log \cos \mathrm{C} - \log \tan \mathrm{AC} + 10 \\ &= 9{\cdot}8260715 - 10{\cdot}1510145 + 10 \\ &= 9{\cdot}6750570, \text{ which is the cotangent of } \\ &\qquad 64^\circ\, 40'\, 34''; \\ \therefore\ \mathrm{BC} &= 64^\circ\, 40'\, 34''. \end{aligned}$$

QUADRANTAL TRIANGLES.

39. Quadrantal triangles can be solved by the same rules as right-angled triangles for using the polar triangle; we see that since one side is a quadrant, and that in the polar triangle $\mathrm{A}' = 180 - a$;

$$\therefore\ \mathrm{A}' = 180^\circ - 90^\circ = 90.$$

In the polar triangle, since $\mathrm{A}' = 90$, we have by the equations, page 10,

$$\cos a' = \cos b' \cos c'$$

$$\sin b' = \sin a' \sin \mathrm{B}' \qquad \sin c' = \sin a' \sin \mathrm{C}'$$

$$\tan b' = \tan a' \cos \mathrm{C}' \qquad \tan c' = \tan a' \cos \mathrm{B}'$$

$$\tan b' = \sin c' \tan B' \qquad \tan c' = \sin b' \tan C'$$
$$\cos B' = \sin C' \cos b' \qquad \cos C' = \sin B' \cos c'$$
$$\cos a' = \cot B' \cot c'.$$

From these by substituting these values

$$a' = 180^\circ - A\,;\; b' = 180^\circ - B\,;\; c' = 180^\circ - C\,;$$
$$A' = 180^\circ - a\,;\; B' = 180^\circ - b\,;\; C' = 180^\circ - c\,;$$

we get these results,

$$\cos A = -\cos B \cos C$$
$$\sin B = \sin A \sin b \qquad \sin C = \sin A \sin b$$
$$\tan B = -\tan A \cos c \qquad \tan C = -\tan A \cos b$$
$$\tan B = \tan b \sin c \qquad \tan C = \sin B \tan c$$
$$\cos b = \sin c \cos B \qquad \cos c = \sin b \cos C$$
$$\cos A = \cot b \cot c$$

Or without using the polar triangle,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ make } a = \text{quadrant},$$

then $\cos a = 0$, and we have

$$\cos A = -\frac{\cos b \cos c}{\sin b \sin c} = -\cot b \cot c\,;$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} = \frac{\cos b}{\sin a}\,;$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} = \frac{\cos c}{\sin b}.$$

From these equations, and the equation $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$, we can deduce all the cases of quadrantal triangles.

Given $A\,B = c = 32^\circ\ 57'\ 6''$ and $A\,C = b = 66^\circ\ 32'$, to find B and A,

$$\cos A = -\cot b \cot c$$
$$\log \cos A = \log \cot b + \log \cot c - 10$$
$$= 10\cdot[illegible]882850 + 9\cdot6376106 - 10$$
$$= 9\cdot8258956, \text{ which is the cosine of } 47^\circ\ 57'\ 16'',$$

but since cos A is negative, A must be greater than 90°.

OBLIQUE-ANGLED TRIANGLES.

40. CASE 1. Given the three sides, viz.

$$\left.\begin{array}{lcl} AB & = & 79^\circ\ 17'\ 14'' \\ BC & = & 110^\circ \\ AC & = & 58^\circ \end{array}\right\}\text{to find the rest.}$$

To find the angle A.

By page 18, $\sin\frac{1}{2}A = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$;

hence we have the following rule :

From half the sum of the three sides subtract each of the two sides which contain the required angle.

Add the log sines of these two remainders, and the complement logs of the sines of the sides which contain the angle.

Half the sum of these four logarithms will give the log sine of half the required angle. Thus:

79° 17′ 14″				
110				
58				
2)247 17 14				
123 38 37	= ½ sum of the three sides.			
79 17 14				
44 21 23	first remainder		log sin =	9,8445513
123 38 37				
58				
65 38 37	second remainder		log sin =	9,9595178
	comp log sin 58°			0,0715795
	comp log sin 79° 17′ 14″			0,0076359
				2)19,8832840

log sin = 60° 57′ 28″ = 9,9416420
2

121 54 56 equals the required angle A.

By a similar operation the angles B and C may be found; but when one angle is known, the other two are easily determined by Art. 13, page 6.

CASE 2. Given the angle $A = 32° 20' 30''$, the side $b = 72° 10' 20''$, and the side $a = 78° 59' 10''$, to find B, C and c.

Here by page 6, $\sin B = \dfrac{\sin A \,.\, \sin b}{\sin a}$

$$\log \sin B = \log \sin A + \log \sin b - \log \sin a$$

$$\begin{array}{rl} \log \sin A = & 9{\cdot}7283269 \\ \log \sin b = & 9{\cdot}9786283 \\ \hline & 19{\cdot}7069552 \\ \log \sin a = & 9{\cdot}9919261 \\ \hline \log \sin B = & 9{\cdot}7150291 \end{array}$$

$$\therefore\ B = 31° 15' 15''.$$

By page 9, equation (16)

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A + B) \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)}$$

$$\log \cot \tfrac{1}{2} C =$$
$$\log \tan \tfrac{1}{2} (A + B) + \log \cos \tfrac{1}{2} (a + b) - \log \cos \tfrac{1}{2} (a - b)$$

$$\begin{array}{rl} \log \tan \frac{1}{2} (A + B) = \log \tan 31° 47' 52'' = & 9{\cdot}7217470 \\ \log \cos \frac{1}{2} (a + b) = \log \cos 75° 34' 45'' = & 9{\cdot}3962727 \\ \hline & 19{\cdot}1180197 \\ \log \cos \frac{1}{2} (a - b) = \log \cos 3° 24' 25'' = & 9{\cdot}9992318 \\ \hline \log \cot \frac{1}{2} C = & 9{\cdot}1187879 \end{array}$$

$$\therefore\ \tfrac{1}{2} C = 82° 30' 39'';$$

$$\text{or } C = 165° 1' 18''.$$

We might find c from the equation

$$\sin c = \sin a \,.\, \frac{\sin C}{\sin A},$$

but we can find it directly from Napier's Analogies.

By page 9, equation (14), we have

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a + b) \,.\, \frac{\cos \frac{1}{2} (A + B)}{\cos \frac{1}{2} (A - B)};$$

$$\therefore \log \tan \tfrac{1}{2} c =$$
$$\log \tan \tfrac{1}{2} (a + b) + \log \cos \tfrac{1}{2} (A + B) - \log \cos \tfrac{1}{2} (A - B)$$

$$\begin{aligned} \log \tan \tfrac{1}{2} (a + b) &= \log \tan 75^\circ\, 34'\, 45'' = 10{\cdot}5898236 \\ \log \cos \tfrac{1}{2} (A + B) &= \log \cos 31^\circ\, 47'\, 52'' = 9{\cdot}9293745 \\ & \hphantom{= \log \cos 31^\circ\, 47'\, 52'' =\ } \overline{20{\cdot}5191981} \\ \log \cos \tfrac{1}{2} (A - B) &= \log \cos 1^\circ\, 5'\, 15'' = 9{\cdot}9999218 \\ \log \tan \tfrac{1}{2} c &= \overline{10{\cdot}5192763} \end{aligned}$$

$$\tfrac{1}{2} c = 73^\circ\, 10'\, 10''$$

$$\therefore c = 146^\circ\, 20'\, 20''.$$

CASE 3. Given $C = 36^\circ\, 45'\, 28''$; $a = 84^\circ\, 14'\, 29''$; $b = 44^\circ\, 13'\, 45''$, the two sides and the included angle, to find A, B, c.

By Napier's Analogies, page 9, equations (16) and (17),

$$\tan \tfrac{1}{2} (A + B) = \cot \tfrac{1}{2} C \,.\, \frac{\cos \frac{1}{2} (a - b)}{\cos \frac{1}{2} (a + b)}$$

$$\text{and } \tan \tfrac{1}{2} (A - B) = \cot \tfrac{1}{2} C \,.\, \frac{\sin \frac{1}{2} (a - b)}{\sin \frac{1}{2} (a + b)}$$

$$\begin{aligned} \tfrac{1}{2} C &= 18^\circ\, 22'\, 44'' \log \cot = 10{\cdot}4785395 \\ \tfrac{1}{2} (a - b) &= 20^\circ\, 0^\circ\, 22'' \log \cos = 9{\cdot}9729690 \\ & \hphantom{= 20^\circ\, 0^\circ\, 22'' \log \cos =\ } \overline{20{\cdot}4515085} \\ \tfrac{1}{2} (a + b) &= 64^\circ\, 14'\, 7'' \log \cos = 9{\cdot}6381663 \end{aligned}$$

$$\therefore \log \tan \tfrac{1}{2} (A + B) = \overline{10{\cdot}8133422}$$

$$\therefore \tfrac{1}{2} (A + B) = 81^\circ\, 15'\, 44''.41.$$

A − B determined.

$$\begin{array}{rlr}
\tfrac{1}{2}\,C = 18^\circ\ 22'\ 44'' & \log\cot = & 10{\cdot}4785395 \\
\tfrac{1}{2}\,(a-b) = 20\ \ \ 0\ \ 22 & \log\sin = & 9{\cdot}5341789 \\
\hline
 & & 20{\cdot}0127184 \\
\tfrac{1}{2}\,(a+b) = 64\ \ 14\ \ \ 7 & \log\sin = & 9{\cdot}9545255 \\
\hline
 & & 10{\cdot}0581929
\end{array}$$

$$\tfrac{1}{2}\,(A-B) = 48^\circ\ 49'\ 38''.$$

A and B determined.

$$\begin{array}{l}
\tfrac{1}{2}(A+B) = 81^\circ\ 15'\ 44''{\cdot}41 \\
\tfrac{1}{2}(A-B) = 48\ \ 49\ \ 38 \\
\hline
\therefore\ A = 130^\circ\ \ 5'\ 22''{\cdot}41 \\
\qquad B = \ 32\ 26\ \ \ \ 6{\cdot}41
\end{array}$$

c determined.

$$\begin{array}{lr}
\log\sin 86^\circ\ 45'\ 28'' = & 9{\cdot}7770158 \\
\log\sin 44\ \ 13\ \ 45\ = & 9{\cdot}8435629 \\
\hline
 & 19{\cdot}6205787 \\
\log\sin 32\ \ 26\ \ \ 6\ = & 9{\cdot}7294422 \\
\hline
\therefore \log\sin c\ = & 9{\cdot}8911365
\end{array}$$

$$\therefore c = 51^\circ\ 6'\ 12''.$$

c may be found directly, without finding A and B, by the following method :—

$$\text{Since } \cos C = \frac{\cos c - \cos a\,.\,\cos b}{\sin a\,.\,\sin b},\ \text{page 5},$$

$$\therefore \cos c = \cos a\,.\,\cos b + \sin a\,.\,\sin b\,.\,\cos C;$$

but $\cos C = 1 - \text{ver.}\sin C$,

$$\therefore \cos c = \cos a\,.\,\cos b + \sin a\,.\,\sin b - \sin a\,.\,\sin b\,.\,\text{ver.}\sin C,$$

$$= \cos(a-b) - \sin a\,.\,\sin b\,.\,\text{ver.}\sin C;$$

$$\therefore 1 - \cos c,\ \text{or}\ 2\sin^2\frac{c}{2} = \text{ver.}\sin(a-b) + \sin a\,.\,\sin b\,.\,\text{ver.}\sin C,$$

$$= \text{ver.}\sin(a-b)\left(1 + \frac{\sin a\,.\,\sin b\,.\,\text{ver.}\sin C}{\text{ver.}\sin(a-b)}\right).$$

$$\text{Let } \tan^2\theta = \frac{\sin a\,.\,\sin b\,.\,\text{ver.}\sin C}{\text{ver}\ \sin(a-b)};$$

which in logarithms is 2 log tan θ =

$$\log \sin a + \log \sin b + \log \text{ver.} \sin C - \log \text{ver.} \sin (a-b) \ldots [a]$$

then $2 \sin^2 \frac{c}{2} = \text{ver.} \sin (a-b) \cdot \sec^2 \theta$, and

$$\log 2 + 2 \log \sin \frac{c}{2} = \log \text{ver.} \sin (a-b) + 2 \log \sec \theta - 10 \ldots [b]$$

c computed independently of A and B.

Finding the auxiliary angle θ by the form [a].

$a = 84° \ 14' \ 29''$ .... sin = 9·9978028
$b = 36 \ \ 13 \ \ 45$ .... sin = 9·8435620
$C = 36 \ \ 45 \ \ 28$ ...., ver. sin = 9·2984762

29·1398419
$a - b = 40 \ \ 0 \ \ 44$ .... ver. sin = 9·3693878

∴ 2 log tan θ .... = 19·7704541
and log tan θ .... = 9·8852270

CASE 4. Given $c = 50° \ 6' \ 20''$; A = 129° 58′ 30″; B = 34° 29′ 30″; to find a, b, C.

By equations (14) and (15), page 9.

$$\tan \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2} c \cdot \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

$$\tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2} c \cdot \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

$$\left.\begin{aligned} \tfrac{1}{2}(A+B) &= 82° \ 14' \\ \tfrac{1}{2}(A-B) &= 47° \ 44' \ 30'' \\ \tfrac{1}{2} c &= 25° \ \ 3' \ 10'' \end{aligned}\right\}$$

$\log \tan \frac{1}{2}(a+b) =$

$$\log \tan \tfrac{1}{2} c + \log \cos \tfrac{1}{2}(A-B) - \log \cos \tfrac{1}{2}(A+B)$$

$\log \tan \frac{1}{2} c = \log \tan 25° \ 3' \ 10''$ = 9·6697162
$\log \cos \frac{1}{2}(A-B) = \log \cos 47° \ 44' \ 30''$ = 9·8276758

19·4973920
$\log \cos \frac{1}{2}(A+B) = \log \cos 82° \ 14'$ = 9·1307812

$\log \tan \frac{1}{2}(a+b)$ = 10·3666108

$$\therefore\ \tfrac{1}{2}(a+b) = 66^\circ\ 44'\ 10''$$

$$\log\tan\tfrac{1}{2}(a-b) = \log\tan\tfrac{1}{2}c + \log\sin\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) - \log\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B})$$

$$\begin{array}{rcr}
\log\tan\tfrac{1}{2}c = \log\tan 25^\circ\ 3'\ 10'' & = & 9\cdot6697162 \\
\log\sin\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \log\sin 47^\circ\ 44'\ 30'' & = & 9\cdot8693023 \\
\hline
 & & 19\cdot5390185 \\
\log\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \log\sin 82^\circ\ 14' & = & 9\cdot9959977 \\
\hline
\log\tan\tfrac{1}{2}(a-b) & = & 9\cdot5430208
\end{array}$$

$$\tfrac{1}{2}(a-b) = 19^\circ\ 14'\ 50''$$

$$\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b) = a$$

$$\tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b) = b$$

$$\begin{array}{r}
66^\circ\ 44'\ 10'' \\
19^\circ\ 14'\ 50'' \\
\hline
85^\circ\ 59'\ \ 0'' \\
\hline
47^\circ\ 29'\ 20'' \\
\hline
\end{array}$$

$$\therefore\ a = 85^\circ\ 59' \text{ and } b = 47^\circ\ 29'\ 20''.$$

To find C.

$$\frac{\sin \mathrm{A}}{\sin a} = \frac{\sin \mathrm{C}}{\sin c}$$

$$\text{or, } \sin \mathrm{C} = \sin c\,.\,\frac{\sin \mathrm{A}}{\sin a}$$

$$\log\sin \mathrm{C} = \log\sin c + \log\sin \mathrm{A} - \log\sin a$$

$$\begin{array}{rcr}
\log\sin \mathrm{A} = \log\sin 129^\circ\ 58'\ 30'' & = & 9\cdot8844129 \\
\log\sin c = \log\sin 50^\circ\ 6'\ 20'' & = & 9\cdot8849241 \\
\hline
 & & 19\cdot7693370 \\
\log\sin a = \log\sin 85^\circ\ 59' & = & 9\cdot9989319 \\
\hline
\log\sin \mathrm{C} & = & 9\cdot7704051
\end{array}$$

$$\therefore\ \mathrm{C} = 36^\circ\ 6'\ 50''.$$

Case 5. Given the angles $A = 70° 39'$; $B = 48° 36'$; $a = 89° 16' 53''$, to find the rest.

By page 21, we have $\sin b = \dfrac{\sin a \sin B}{\sin A}$

$$\log \sin b = \log \sin a + \log \sin B - \log \sin A$$

$$\begin{array}{lr}
\log \sin a = \log \sin 89° 16' 53'' = & 9{\cdot}9999658 \\
\log \sin B = \log \sin 48° 36' \quad = & 9{\cdot}8751256 \\
\hline
 & 19{\cdot}8750914 \\
\log \sin A = \log \sin 70° 39' \quad = & 9{\cdot}9747475 \\
\hline
\log \sin b \quad = & 9{\cdot}9003439
\end{array}$$

$$\therefore\ b = 52° 39' 4''$$

$\sin b = \sin(180 - b) = \sin 127° 20' 56''$; but since $A > B$, a must be greater than b, hence b cannot be $127° 20' 56''$.

To find c.

By Napier's Analogies

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2}(a + b) \cdot \frac{\cos \tfrac{1}{2}(A + B)}{\cos \tfrac{1}{2}(A - B)}$$

$\log \tan \frac{1}{2} c =$

$$\log \tan \tfrac{1}{2}(a + b) + \log \cos \tfrac{1}{2}(A + B) - \log \cos \tfrac{1}{2}(A - B)$$

$$\begin{array}{lr}
\log \tan \tfrac{1}{2}(a + b) = \log \tan 70° 57' 59'' = & 10{\cdot}4622011 \\
\log \cos \tfrac{1}{2}(A + B) = \log \cos 59° 37' 30'' = & 9{\cdot}7038563 \\
\hline
 & 20{\cdot}1660574 \\
\log \cos \tfrac{1}{2}(A - B) = \log \cos 11° 1' 30'' = & 9{\cdot}9919097 \\
\hline
\log \tan \tfrac{1}{2} c \quad = & 10{\cdot}1741477
\end{array}$$

$$\therefore\ \tfrac{1}{2} c = 56° 11' 29'',$$

$$\text{or } c = 112° 22' 58''.$$

By equation (16), page 9, we have

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A + B) \frac{\cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}(a - b)}$$

log cot ½ C log =

$$\log \tan \tfrac{1}{2}(\text{A} + \text{B}) + \log \cos \tfrac{1}{2}(a + b) - \log \cos \tfrac{1}{2}(a - b)$$

$$\begin{array}{rcr}
\log \tan \tfrac{1}{2}(\text{A} + \text{B}) = \log \tan 59^\circ\, 37'\, 30'' & = & 10{\cdot}2320208 \\
\log \cos \tfrac{1}{2}(a + b) = \log \cos 70^\circ\, 57'\, 59'' & = & \underline{9{\cdot}5133811} \\
 & & 19{\cdot}7454019 \\
\log \cos \tfrac{1}{2}(a - b) = \log \cos 18^\circ\, 18'\, 54'' & = & \underline{9{\cdot}9774233} \\
\log \cot \tfrac{1}{2}\text{C} & = & 9{\cdot}7679786
\end{array}$$

$$\therefore \tfrac{1}{2}\text{C} = 59^\circ\, 37'\, 30'',$$

$$\text{or } \text{C} = 119^\circ\, 15'.$$

CASE 6. Given the three angles,

$$\left.\begin{array}{lcl}
\text{angle A} & = & 120^\circ\, 51'\, 56'' \\
\text{angle B} & = & 50 \\
\text{angle C} & = & 62\ \ 34\ \ \ 6
\end{array}\right\}\ \text{to find the rest.}$$

By page 22, $\cos \frac{1}{2} a = \sqrt{\dfrac{\cos(\text{S} - \text{B}) \cos(\text{S} - \text{C})}{\sin \text{B} \sin \text{C}}}$;

hence the following rule.

To find the side BC.

From half the sum of the three angles take each of the angles next the required side.

Add the log cosines of these two remainders, and the comp. log of the sines of each of the adjoining angles.

Half the sum of these four logarithms will give the cosine of half the required side; thus

121° 54′ 56″
50
62 34 6

2) 234 29 2

117 14 31
62 34 6

54 40 25 first rem. cos 9,7621032
67 14 31 second rem. cos 9,5875321
comp. log sin 50° 0,1157460
comp. log sin 62° 34′ 6″ 0,0518018

2) 19,5171831

cosine 55° 9,7585915
2

110° = the required side BC.

By the same method the other sides may be found; but one side being known (with the angles) the rest are most readily found by Art. 14, page 6.

CHAPTER III.

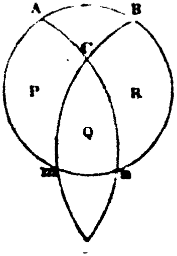
41. THE surface of the sphere included between the arcs D M, D N is proportional to the angle N D M or the arc M N. See fig. page 1.

If the circumference be divided into equal parts as M N, and great circles be drawn from D through the points M, N, the portions of the surface, such as N D M, are all similar and equal, hence if F M contains N M, n times, or if F M $= n$ times N M, the surface F D M will be n times N D M.

When D M coincides with D G, the angle F D G or its measure F M G $= 180$; hence if S $=$ whole surface of the sphere, and if A be the angle N D M which is measured by the arc N M, the surface N D M $= \frac{S}{4} \cdot \frac{A}{180}$, but S $=$ area of 4 great circles of the sphere. Hall's Diff. Cal., page 370.

$$\therefore \ S = 4\,\pi\,r^2 = 4\,\pi \text{ when radius is unity,}$$

$$\text{or } \frac{S}{4} = \pi = 180.$$



The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

Let the triangle be A B C, a, b, c, representing the magnitudes of the angles at A, B, C; let P$=$surface B C m B, Q$=$$m$ C n m, R$=$A C n A; produce the arcs C m, C n, till they meet at e (which will be on the hemisphere opposite to that represented by A B m n A), then each of the angles at C and e equals the angle of the planes in which the arcs C m e, C n e, lie; therefore the angles at C and e are equal.

Again, the semicircles $A\,C\,m$, $C\,m\,e$; $B\,C\,n$, $C\,n\,e$ are equal; or, $A\,C + C\,m = C\,m + m\,e$, and $\therefore A\,C = m\,e$, and $B\,C = n\,e$; and the triangle $m\,e\,n =$ the triangle $A\,B\,C$; let $x =$ its area, then, by last article,

$$x + P = \frac{S}{2} \cdot \frac{a}{180}$$

$$x + Q = \frac{S}{2} \cdot \frac{c}{180}, \text{ and } x + P + Q + R = \frac{S}{2}$$

$$x + R = \frac{S}{2} \cdot \frac{b}{180};$$

consequently, by addition,

$$2\,x + (x + P + Q + R) \text{ or } 2\,x + \frac{S}{2} = \frac{S}{2} \cdot \left(\frac{a+b+c}{180}\right);$$

$$\therefore x = \frac{S}{4\,(180)}(a+b+c) - \frac{S}{4} = a + b + c - 180^\circ;$$

$$\text{or } r^2\,(a + b + c - 180^\circ).$$

Hence the area of a spherical triangle is equal to the excess of the sum of its three angles above two right angles, which is called the *spherical excess.*

The late Professor Woodhouse, in his able work on Trigonometry, observes that—"This expression for the value of the area was merely a speculative truth, and continued barren for more than 150 years, till 1787, when General Roy employed it in correcting the spherical angles of observation made in the great Trigonometrical Survey."

In a biographical sketch of the life of Isaac Dalby, late Professor of Mathematics at the Royal Military College, Sandhurst, in Leybourne's Mathematical Repository, it is stated that he had sent some years previously to his death an account of the principal events of his life after reaching maturity. The following is a quotation from himself given in the above-named excellent periodical :—

"General Roy's account of this measurement is in the Philosophical Transactions; but it is not altogether what it ought to have been. His description of the apparatus, detail of occurrences, &c., are all well enough; but he should not have meddled with the mathematical part, *for his knowledge did not extend beyond Plane Trigonometry.* I drew up the computations in that form which I thought the most proper for publication, but he was continually making alterations. *He did not even understand the rule I made use of for finding the excess of the sum of the three angles of a spherical triangle above 180° (which since that time has been quoted as General Roy's theorem), and would not insert it until he had consulted the Hon. Henry Cavendish.* For conducting the business in the field, however, few persons could have been better qualified than the General.

"I believe he was the best topographer in England, and knew the situation of every barrow, cairn, and hillock in Great Britain. He had something of an observatory in the upper part of his dwelling, and could regulate a clock or watch by means of transits. In fact, he was ready enough at calculations which depended merely on the use of the tables. *But the rules which he published for measuring the heights of the barometer all came from Mr. Ramsden.*"

A note is given to this extract in the Repository, which is as follows:—"It is not until very recently that Mr. Dalby has had justice done him with regard to this ingenious rule. At page 138 of the new edition of Vol. III. of Hutton's Course of Mathematics, published by Dr. Gregory in 1827, we find this note:—'This is commonly called General Roy's rule, and given by him in the Philosophical Transactions for 1790, p. 171; it is, however, due to the late Mr. Isaac Dalby, who was then General Roy's assistant in the Trigonometrical Survey, and for several years the entire conductor of the mathematical department.'"

FURTHER DEVELOPMENTS CONCERNING THE SPHERICAL EXCESS.

42. Let the radius of the sphere be unity, π the semicircumference of a great circle; a, b, c, the three sides of a spherical triangle; A, B, C, the arcs of a great circle that measure the opposite angles.

Let the spherical excess $A + B + C - \pi = S$.

The area of the spherical triangle is equal to the arc S multiplied by the radius, and is therefore represented by S.

Now, by Napier's Analogies,

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \,.\, \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}$$

$$\tan \left\{ \tfrac{1}{2}(A + B) + \tfrac{1}{2} C \right\} = \frac{\tan \frac{1}{2}(A + B) + \tan \frac{1}{2} C}{1 - \tan \frac{1}{2}(A + B) \tan \frac{1}{2} C}$$

$$= \frac{\cot \frac{1}{2} C \,.\, \dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} + \tan \frac{1}{2} C}{1 - \cot \frac{1}{2} C \,.\, \dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \,.\, \tan \frac{1}{2} C}$$

$$= \frac{\cot \frac{1}{2} C \,.\, \dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} + \tan \frac{1}{2} C}{1 - \dfrac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}}$$

$$= \frac{\cot \frac{1}{2} C \,.\, \cos \frac{1}{2}(a - b) + \tan \frac{1}{2} C \,.\, \cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a + b) - \cos \frac{1}{2}(a - b)}$$

$$\frac{\dfrac{1 + \cos C}{\sin C} \cos \frac{1}{2}(a - b) + \dfrac{1 - \cos C}{\sin C} \cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a + b) - \cos \frac{1}{2}(a - b)} =$$

(by expanding the cosines and reducing)

$$\frac{1}{\sin C} \,.\, \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b + \sin \frac{1}{2} a \sin \frac{1}{2} b \cos C}{- \sin \frac{1}{2} a \sin \frac{1}{2} b}$$

$$= \frac{- \cot \frac{1}{2} a \cot \frac{1}{2} b - \cos C}{\sin C}$$

but $\tan \frac{1}{2}(A + B + C) = \tan \frac{1}{2}(S + 180) = - \cot \frac{1}{2} S$;

$$\therefore \cot \tfrac{1}{2} S = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}.$$

This equation, which is very simple, enables us to find the area of a spherical triangle when the two sides and the included angle are given.

To find the area of a spherical triangle in terms of the three sides,

$$\cot \tfrac{1}{2} S = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C}$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

$$\text{and } \cot \tfrac{1}{2} a = \frac{1 + \cos a}{\sin a}, \quad \cot \tfrac{1}{2} b = \frac{1 + \cos b}{\sin b};$$

$$\therefore \cos C + \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b}$$

$$1 + \cos C = \frac{\cos c - \cos (a+b)}{\sin a \sin b} = \frac{2 \sin \frac{+b+c}{2} \sin \frac{a+b-c}{2}}{\sin a \sin b}$$

$$1 - \cos C = \frac{\cos (a-b) - \cos c}{\sin a \sin b} = \frac{\left(\frac{2 \sin a+b+c)}{2}\right) \sin \frac{b+c-a}{2}}{\sin a \sin b}$$

Multiplying these two equations and extracting the root,

$$\sin C = 2 \frac{\sqrt{\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}}}{\sin a \sin b}$$

By substituting these values we have

$$\cot \tfrac{1}{2} S = 2 \frac{1 + \cos a + \cos b + \cos c}{\sqrt{\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+a-b}{2}}}$$

This solves the problem, but it can be put into a simpler form, and one that is adapted to logarithmic computation.

Resuming the formula

$$\cot \tfrac{1}{2} S = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C},$$

$$1 + \cot^2 \tfrac{1}{2} S = \frac{1}{\sin^2 \tfrac{1}{2} S} = \frac{\cot^2 \tfrac{1}{2} a \cot^2 \tfrac{1}{2} b + 2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C + 1}{\sin^2 C},$$

by multiplying both sides of the value of cos C by 2 cot ½ *a* cot ½ *b*, we have

$$2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C = \frac{\cos c - \cos a \cos b}{2 \sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b},$$

putting in the numerator for cos *c*, cos *a*, cos *b*, their values $1 - 2 \sin^2 \tfrac{1}{2} c$, $1 - 2 \sin^2 \tfrac{1}{2} a$, $1 - 2 \sin^2 \tfrac{1}{2} b$, we shall have

$$2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C = \frac{\sin^2 \tfrac{1}{2} a + \sin^2 \tfrac{1}{2} b - \sin^2 \tfrac{1}{2} c}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b} - 2.$$

$$\text{Also, } \cot^2 \tfrac{1}{2} a \cot^2 \tfrac{1}{2} b = \frac{1 - \sin^2 \tfrac{1}{2} a}{\sin^2 \tfrac{1}{2} a} \cdot \frac{1 - \sin^2 \tfrac{1}{2} b}{\sin^2 \tfrac{1}{2} b} =$$
$$\frac{1 - \sin^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} b}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b} + 1.$$

Substituting all these values, we have

$$\frac{1}{\sin^2 \tfrac{1}{2} S} = \frac{1 - \sin^2 \tfrac{1}{2} c}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b \sin^2 C}, \text{ or}$$

$$\sin \tfrac{1}{2} S = \frac{\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin C}{\cos \tfrac{1}{2} c}.$$

Substituting for sin C its value, we have

$\sin\frac{1}{2}S =$

$$\frac{\sqrt{\left(\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}\right)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$$

which expression is adapted to logarithmic computation.

If we multiply this equation by $\cot\frac{1}{2}S$ we have

$$\cos\frac{1}{2}S = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} =$$

$$\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c};$$

From this we have,

$$\frac{1-\cos\frac{1}{2}S}{\sin\frac{1}{2}S} \text{ or } \tan\frac{1}{4}S =$$

$$\frac{1-\cos^2\frac{1}{2}a-\cos^2\frac{1}{2}b-\cos^2\frac{1}{2}c+2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}{\sqrt{\left\{\sin\frac{a+b+c}{2}.\sin\frac{a+b-c}{2}.\sin\frac{a+c-b}{2}.\sin\frac{b+c-a}{2}\right\}}}$$

The numerator of this expression can be put under the form $(1-\cos^2\frac{1}{2}a)(1-\cos^2\frac{1}{2}b)-(\cos\frac{1}{2}a\cos\frac{1}{2}b-\cos\frac{1}{2}c)^2$ which may be decomposed into the factors

$$\sin\frac{1}{2}a\sin\frac{1}{2}b+\cos\frac{1}{2}a\cos\frac{1}{2}b-\cos\frac{1}{2}c \text{ and}$$

$$\sin\frac{1}{2}a\sin\frac{1}{2}b-\cos\frac{1}{2}a\cos\frac{1}{2}b+\cos\frac{1}{2}c.$$

These reduce ultimately, the first to $\cos(\frac{1}{2}a-\frac{1}{2}b)-\cos\frac{1}{2}c=$

$$2\sin\frac{a+c-b}{4}\sin\frac{b+c-a}{4},$$

the second to $\cos\frac{1}{2}c-\cos(\frac{1}{2}a+\frac{1}{2}b)=$

$$2\sin\frac{a+b+c}{4}\sin\frac{a+b-c}{4};$$

$$\therefore \tan \tfrac{1}{4} S =$$

$$\frac{4 \sin \dfrac{a+b+c}{4} \sin \dfrac{a+b-c}{4} \sin \dfrac{a+c-b}{4} \sin \dfrac{b+c-a}{4}}{\sqrt{\left\{\sin \dfrac{a+b+c}{2} \sin \dfrac{a+b-c}{2} \sin \dfrac{a+c-b}{2} \sin \dfrac{b+c-a}{2}\right\}}}$$

$$\therefore \tan \tfrac{1}{4} S =$$

$$\sqrt{\tan \frac{a+b+c}{4} \tan \frac{a+b-c}{4} \tan \frac{a+c-b}{4} \cdot \tan \frac{b+c-a}{4}}.$$

This elegant formula is due to Simon Lhuillier. See Legendre's Geometry, page 319

GIRARD'S THEOREM.

43. By page 45,

$a + b + c - 180 = \frac{180}{\pi r^2} x$, or reducing a, b, c to seconds,

the excess in seconds $= \frac{180^\circ \times 60 \times 60}{\pi r^2} x$.

Now, on the earth's surface, the length of 1° taking a mean measurement $= (60859 \cdot 1) \times 6$ feet, and an arc $=$ radius $= \frac{360}{2\pi}$;

$$\therefore (60859 \cdot 1) \times 6 \times \frac{360}{2\pi} = \text{radius of the earth in feet};$$

$\therefore$ the excess in seconds $= x \cdot \frac{2\pi}{360} \cdot \frac{60 \times 60}{6^2 \times (60859 \cdot 1)^2} =$

$\frac{2\pi \times 10}{36 \times (60859 \cdot 1)^2} \cdot x.$

$$\log . \text{excess} =$$
$$\log x - \{2 (\log 6 + \log 60859 \cdot 1) - \log 2\pi \times 10\}$$
$$= \log x - (11 \cdot 1249536 - 1 \cdot 7981799)$$
$$= \log x - 9 \cdot 3267737.$$

Hence the following rule:—

From the logarithm of the area of the triangle, considered as a plane one, in feet, subtract the constant logarithm 9·3267737, and the remainder is the logarithm of the excess above 180°, in seconds nearly.

		Observed angles.		
Ex. *Hanger Hill Tower*	(a)	42°	2′	32″
Hampton Poor-house	(b)	67	55	39
King's Arbour	(c)	70	1	48

Distance from (a) to (b) = 38461·12 feet,
from (a) to (c) = 24704·7.

Taking the distance from (a) to (c) for the base of the triangle, the perpendicular on the base will be 38461·12 × sin 42° 2′ 32″, and therefore the area of the triangle

$$= \frac{\text{base} \times \text{perpendicular}}{2}$$

$$= 24704{\cdot}7 \times 19230{\cdot}56 \times \sin 42^\circ\ 2'\ 32'',$$

$$\log \text{area} = \log 24704{\cdot}7 + \log 19230{\cdot}56$$

$$+ \log \sin 42^\circ\ 2'\ 32'' - 10$$

$$= 4{\cdot}3927761 + 4{\cdot}2839906 + 9{\cdot}8258661$$

$$= 8{\cdot}5026328 = \text{logarithm of the area in feet};$$

$$\text{hence, } 8{\cdot}5026328 - 9{\cdot}3267737 = -\ 1{\cdot}1758591;$$

the corresponding natural number is ·14992, the spherical excess in seconds.

LEGENDRE'S SOLUTION OF SPHERICAL TRIANGLES WHOSE SIDES ARE VERY SMALL COMPARED WITH THE RADIUS OF THE SPHERE.

44. When the sides a, b, c, are very small with respect to the radius of the sphere, the proposed triangle is very little different from a rectilinear triangle, and, considering it as such, we can have a first solution approximately, but we neglect in this manner the excess of the sum of the three angles above 180°. To have a solution more approximate, we must take

into account this excess, and this we can do very easily by means of a general principle which we proceed to demonstrate.

Let r be the radius of the sphere upon which the triangle is situated, and if we imagine a similar triangle upon the sphere whose radius is unity, the sides of this triangle will be $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; and we shall have $\cos A =$

$$\frac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}};$$

but since r is very great with respect to a, b, c, we shall have approximately,

$$\cos\frac{a}{r} = 1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4r^4};$$

$$\cos\frac{b}{r} = 1 - \frac{b^2}{2r^2} + \frac{b^4}{2.3.4r^4},$$

$$\cos\frac{c}{r} = 1 - \frac{c^2}{2r^2} + \frac{c^4}{2.3.4r^4},$$

$$\sin\frac{b}{r} = \frac{b}{r} - \frac{b^3}{2.3r^3};$$

$$\sin\frac{c}{r} = \frac{c}{r} - \frac{c^3}{2.3r^3}$$

Substituting these values in the above equation,

$$\cos A = \frac{\frac{b^2+c^2-a^2}{2r^2} + \frac{a^4-b^4-c^4}{24r^4} - \frac{b^2c^2}{4r^4}}{\frac{bc}{r^2}\left(1 - \frac{b^2}{6r^2} - \frac{c^2}{6r^2}\right)}$$

Multiply numerator and denominator by $1 + \frac{b^2+c^2}{6r^2}$ and reducing

$$\cos A = \frac{b^2+c^2-a^2}{2\,b\,c} + \frac{a^4+b^4+c^4-2\,a^2\,b^2-2\,a^2\,c^2-2\,b^2\,c^2}{24\,b\,c\,r^2}$$

Let A′ be the angle opposite to the side a in the rectilinear triangle of which the sides are equal in length to the arcs a, b, c, we shall have

$$\cos A' = \frac{b^2+c^2-a^2}{2\,b\,c}, \text{ and}$$

$$4\,b^2\,c^2\sin^2 A' = 2\,a^2\,b^2+2\,a^2\,c^2+2\,b^2\,c^2-a^4-b^4-c^4;$$

therefore $\cos A = \cos A' - \frac{b\,c}{6\,r^2}\,.\,\sin^2 A'$

let $A = A' + x$, we shall have rejecting the square of x,

$$\cos A = \cos A' - x\sin A'.$$

from whence we have $x = \frac{b\,c}{6\,r^2}\sin A'$,

and since x is of the second order with respect to $\frac{b}{r}$ and $\frac{c}{r}$, it follows that the result is exact to quantities of the fourth order, we shall then have

$$A = A' + \frac{b\,c}{6\,r^2}\sin A';$$

but $\frac{1}{2}\,b\,c\sin A'$ = the area of the rectilinear triangle, of which the three sides are a, b, c, do not differ sensibly from those of the proposed spherical triangle. Then, if either area be called α, we shall have

$$A = A' + \frac{\alpha}{3\,r^2}, \text{ or } A' = A - \frac{\alpha}{3\,r^2}.$$

Similarly, $B' = B - \frac{\alpha}{3\,r^2}$; $C' = C - \frac{\alpha}{3\,r^2}$,

hence there results,

$$A' + B' + C' \text{ or } 180 = A + B + C - \frac{\alpha}{r^2},$$

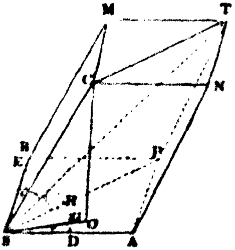
we can then consider $\frac{\alpha}{r^2}$ as the excess of the three angles of the spherical triangle above two right angles.

Hence we have the following rule:—

A spherical triangle being proposed, of which the sides are very small with respect to the radius of the earth, if from each of its angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, the sides of which are equal in length to those of the proposed spherical triangle, or in other terms:—The spherical triangle, whose sides are nearly rectilinear, of which the angles are A, B, C, and the opposite sides a, b, c, answer always to a rectilinear triangle whose sides are of the same length, a, b, c, and of which the opposite angles are $A - \frac{1}{3}\epsilon$; $B - \frac{1}{3}\epsilon$; $C - \frac{1}{3}\epsilon$; ϵ being the excess of the sum of the angles of the spherical triangle proposed above two right angles.

The excess ϵ, or $\frac{\alpha}{r^2}$, which is proportional to the area of the triangle, can always be calculated *à priori*, by the given parts of the spherical triangle considered as rectilinear. If the two sides, b, c, and the included angle A, are given, we shall have the area $\alpha = \frac{1}{2} bc \sin A$; if we have given the side a, and the two adjacent angles B, C, we shall have the area

$$\alpha = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B + C)}.$$



45. Given the three edges of a parallelopiped, and the angles between them, to find the solidity.

Let the edges $SA = f$, $SB = g$, $SC = h$, and the contained angles $ASB = \alpha$, $ASC = \beta$, $BSC = \gamma$, if from the point C we let fall C O perpendicularly on the plane ASB then in right-angled triangle CSO; $CO = CS \sin CSO = h \sin CSO$, besides the surface of the parallelogram ASBP $= f\,g \sin \alpha$. Therefore, if we call S the solidity of the parallelopiped ST, we shall have $S = f.\,g.\,h.\ \sin \alpha \sin \mathrm{CSO}$. We now proceed to find sin CSO. From the point S as a centre and radius unity, describe a spherical surface meeting the right lines SA, SB, SC, SO, in the points D, E, F, G, we shall have a triangle DEF, in which the arc FG is perpendicular to ED, since the plane CSO is perpendicular to ASB. Now, in the triangle DEF, where the three sides, $\mathrm{DE} = \alpha$, $\mathrm{DF} = \beta$, $\mathrm{EF} = \gamma$, we have

$$\cos \mathrm{E} = \frac{\cos \beta - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}, \text{ and}$$

$$\sin \mathrm{E} = \frac{\sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}}{\sin \alpha \sin \gamma}$$

Then in the right-angled triangle EFG, sin GF or sin CSO $=$ sin E sin EF $= \sin \gamma \sin \mathrm{E}$.

$$\therefore S = f.\,g.\,h.\ \sin \alpha \sin \gamma \sin \mathrm{E} =$$

$$f.\,g.\,h.\ \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}.$$

The expression under the radical is composed of the two factors, $\sin \alpha \sin \gamma + \cos \beta - \cos \alpha \cos \gamma$, and $\sin \alpha \sin \gamma - \cos \beta + \cos \alpha \cos \gamma$, the first $= \cos \beta - \cos(\alpha + \gamma) =$

$$\frac{2 \sin \alpha + \beta + \gamma}{2} \cdot \sin \frac{\alpha + \gamma - \beta}{2};$$

the second $= \cos(\alpha - \gamma) - \cos \beta =$

$$2 \sin \frac{\alpha + \beta - \gamma}{2} \cdot \sin \frac{\beta + \gamma - \alpha}{2};$$

therefore, S $=$

$$2fgh\sqrt{\sin \frac{\alpha+\beta+\gamma}{2} \cdot \sin \frac{\alpha+\beta-\gamma}{2} \cdot \sin \frac{\alpha+\gamma-\beta}{2} \cdot \sin \frac{\beta+\gamma-\alpha}{2}}$$

46. The same things being given as in the above to find the diagonal.

Let the diagonal of the base $SP = z$, and the required diagonal $ST = u$, the triangle ASP, in which $\cos SAP = -\cos\alpha$, we have $z^2 = f^2 + g^2 + 2fg\cos\alpha$, in like manner in the triangle TSP, in which $\cos TPS = -\cos CSP$; $u^2 = z^2 + h^2 + 2hz\cos CSP$.

We must now find $\cos CSP$, or of the arc FH.

Now, in the spherical triangle EFH, we have $\cos FH = \cos EF \cos EH + \sin EF \sin EH \cos E$, substituting the values $FF = \gamma$ and $\cos E = \dfrac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}$ it becomes

$$\cos FH = \cos\gamma\cos EH + \frac{\sin EH}{\sin\alpha}(\cos\beta - \cos\alpha\cos\gamma) =$$

$$\frac{\sin EH\cos\beta}{\sin\alpha} + \frac{\sin(\alpha - EH)\cos\gamma}{\sin\alpha}$$

$$\frac{\sin EH\cos\beta + \sin DH\cos\gamma}{\sin\alpha}.$$

Therefore $2hz\cos FH$, or $2hz\cos CSP =$

$$2h\cos\beta\,.\,\frac{z\sin EH}{\sin\alpha} + 2h\cos\gamma\,.\,\frac{z\sin DH}{\sin\alpha};$$

but in the triangle BSP we have

$$BP = \frac{SP\sin BSP}{\sin SBP} \text{ and } BS = \frac{SP\sin BPS}{\sin SBP};$$

$$\text{which gives } \frac{z\sin EH}{\sin\alpha} = f \text{ and } \frac{z\sin DH}{\sin\alpha} = g;$$

$$\therefore\ 2hz\cos CSP = 2fh\cos\beta + 2gh\cos\gamma.$$

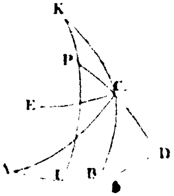
Hence the square of the required diagonal

$$u^2 = f^2 + g^2 + h^2 + 2fg\cos\alpha + 2fh\cos\beta + 2gh\cos\gamma.$$

47. To determine a line on the surface of a sphere on which the vertices of all triangles of the same base and surface are situated.

Let ABC, be a spherical triangle, (one of those on the common base AB, $= c$); and the given surface $A + B + C - \pi = S$. Let IPK be an indefinite perpendicular on the middle of AB, having taken IP $=$ a quadrant, P will be the pole of the arc AB, and the arc CD, drawn through the points P C, will be perpendicular to AB. Let ID $= p$, CD $= q$, the right-angled triangles, ACD, BCD, in which AC $= b$, BC $= a$, AD $= p + \frac{1}{2}c$, BD $= p - \frac{1}{2}c$, will give $\cos a = \cos q.\ \cos(p - \frac{1}{2}c)$, $\cos b = \cos q \cos(p + \frac{1}{2}c)$. But it was found, page 48, that

$$\cot \tfrac{1}{2} S = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b \sin C}.$$



Substituting in this formula the values $\cos a + \cos b = 2 \cos q \cos p \cos \frac{1}{2} c$, $1 - \cos c = 2 \cos^2 \frac{1}{2} c$, $\sin b \sin C = \sin c \sin B = 2 \sin \frac{1}{2} c \cos \frac{1}{2} c \sin B$; there results,

$$\cot \tfrac{1}{2} S = \frac{\cos \frac{1}{2} c + \cos p \cos q}{\sin a \sin \frac{1}{2} c \sin B}.$$

Again, from the right-angled triangle BCD, $\sin a \sin B = \sin q$;

$$\therefore \cot \tfrac{1}{2} S = \frac{\cos \frac{1}{2} c + \cos p \,.\, \cos q}{\sin \frac{1}{2} c \sin q},$$

or, $\cos p \cos q = \cot \frac{1}{2} S \sin \frac{1}{2} c \sin q - \cos \frac{1}{2} c$. This is the relation between p and q which will determine the locus of all the points C.

Produce IP to K, let PK $= x$. Join KC, and let KC $= y$; in the triangle PKC where we have PC $= \frac{1}{2}\pi - q$, the angle KPC $= \pi - p$, the side KC will be found by the formula

$$\cos KC = \cos KPC \sin PK \sin PC + \cos PK \cos PC, \text{ or}$$

$$\cos y = \sin q \cos x - \sin x \cos p \cos q.$$

Substituting this instead of $\cos p \cos q$ the value

$$\cos \tfrac{1}{2} S \sin \tfrac{1}{2} c \sin q - \cos \tfrac{1}{2} c, \text{ there results}$$

$$\cos y = \sin x \cos \tfrac{1}{2} c + \sin q (\cos x - \sin x \cot \tfrac{1}{2} S \sin \tfrac{1}{2} c).$$

In which, if we take $\cos x - \sin x \cot \frac{1}{2} S \sin \frac{1}{2} c = 0$, or $\cot \frac{1}{2} S \sin \frac{1}{2} c = \cot x$, there will result $\cos y = \sin x \cos \frac{1}{2} c$, and thus a constant value of y is determined.

Therefore, if after having drawn the arc IP perpendicular to the middle of the base AB, and beyond the pole the part PK such that $\cot PK = \cot \frac{1}{2} S \sin \frac{1}{2} c$, all the vertices of the triangles on the same base c, and of the same surface S, will be situated on the small circle described from K as a pole at the distance KC, such that $\cos KC = \sin PK \cos \frac{1}{2} c$. This is Lexell's theorem.

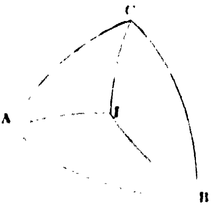
48. Given the three sides, $BC = a$, $AC = b$, $AB = c$, to find the position of the point I, the pole of the circle circumscribing the triangle ABC.

Let the angle $ACI = x$, and the arc $AI = CI = BI = \varphi$; in the triangles CAI, CBI, we shall have by the equation—

$$\cos x = \frac{\cos \varphi - \cos b \cos \varphi}{\sin b \sin \varphi} =$$

$$\frac{1 - \cos b}{\sin b} \cot \varphi =$$

$$\frac{\sin b}{1 + \cos b} \cot \varphi.$$



$$\cos (C - x) = \frac{1 - \cos a}{\sin a} \cot \varphi; \text{ therefore } \frac{\cos (C - x)}{\cos x};$$

$$\text{or, } \cos C + \sin C \tan x = \frac{(1 + \cos b)(1 - \cos a)}{\sin a \sin b}.$$

Substituting, in this equation, the values of cos C and sin C, in terms of the sides a, b, c, and putting for the sake of abridgment, $M = \sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}$, we have $\tan x = \dfrac{1 + \cos b - \cos c - \cos a}{M}$, which determines the angle ACI. From the isosceles triangles ACI, ABI, BCI, we have $ACI = \frac{1}{2}(C + A - B)$; and, in the same manner, $BCI = \frac{1}{2}(B + C - A)$; $BAI = \frac{1}{2}(A + B - C)$.

From which results these remarkable formulæ,

$$\tan \tfrac{1}{2}(A + C - B) = \frac{1 + \cos b - \cos a - \cos c}{M}$$

$$\tan \tfrac{1}{2}(B+C-A) = \frac{1+\cos a-\cos b-\cos c}{M}$$

$$\tan \tfrac{1}{2}(A+B-C) = \frac{1+\cos c-\cos a-\cos b}{M}.$$

To which we may add that which gives $\cot \tfrac{1}{2}$ S, and whic can be put under the form

$$\tan \tfrac{1}{2}(A+B+C) = \frac{-1-\cos a-\cos b-\cos c}{M}.$$

From the value of the tangent of x, already found, we hav

$$1+\tan^2 x = \frac{1}{\cos^2 x} = \frac{2(1+\cos b)(1-\cos c)(1-\cos a)}{M^2} =$$

$$\frac{16\cos^2 \frac{1}{2} b \sin^2 \frac{1}{2} c \sin^2 \frac{1}{2} a}{M^2};$$

$$\therefore \frac{1}{\cos x} = \frac{4\cos \frac{1}{2} b \sin \frac{1}{2} c \sin \frac{1}{2} a}{M}, \text{ but from the equation}$$

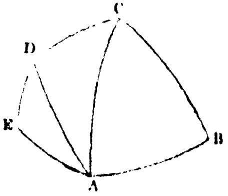
$$\cos x = \frac{1-\cos b}{\sin b}\cot \varphi = \tan \tfrac{1}{2} b \cot \varphi, \text{ we find}$$

$$\tan \varphi = \frac{\tan \frac{1}{2} b}{\cos x}; \quad \therefore \tan \varphi = \frac{4\sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{M}$$

$$= \frac{2\sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\left(\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}\right)}}.$$

49. The surface of a spherical polygon is measured by the sum of the angles, minus the product of two right angles, and the number of sides of the polygon, minus 2.

From A draw the arcs AC and AD, the angles of the polygon; it will then be divided into as many triangles, minus two, as the figure has sides; but the surface of each triangle is measured by the sum of the angles, minus two right angles; and it is clear, that the sum of all the angles of the triangles is equal to the sum of all the angles of the polygon.



Therefore the surface of the polygon is equal to the sum of the angles diminished by as many times two right angles as the figure has sides, minus two.

Thus if S = sum of the angles of a spherical polygon, n = the number of its sides, then the surface of the polygon is $S - n\pi + 2\pi = S - 2(n - 2)$ or $S - 2n + 4$, when the right angle is taken equal to unity.

POLYHEDRONS.

50. If S be the number of solid angles of a polyhedron, H the number of faces, A the number of its edges, then

$$S + H = A + 2.$$

Take a point within the polyhedron, and from which draw lines to all the angular points of the polyhedron: imagine from this point, as a centre, we describe a spherical surface which meets all these lines in as many points, then join these points by arcs of great circles, in such a manner as to form, upon the surface of the sphere, the same number of polygons as there are faces of the polyhedron.

Let ABCDE be one of these polygons, and n the number of its sides, its surface by the last article will be $S - 2n + 4$; S being the sum of the angles A, B, C, D, E. Similarly if we find the value of each of the other spherical polygons, and add them all together, we conclude that their sum or the surface of the sphere which is represented by 8, is equal to the sum of all the angles of the polygons, less twice the number of their sides, plus four times the number of faces.

Now as all the angles that meet at the same point, A, is equal to four right angles, the sum of all the angles of the polygons is equal to four times the number of solid angles, it is therefore equal to 4 S. Then double the number of sides AB, BC, CD, &c., is equal to four times the number of edges, or equal to 4 A, since the same edge serves for two faces;

$$\therefore 8 = 4S - 4A + 4H; \text{ or}$$
$$2 = S - A + H; \text{ or } S + H = A + 2.$$

[See *Legendre's Geometry*, pp. 228, 229.]

Cor. It follows that the sum of all the plane angles which form the solid angles of a regular polyhedron, is equal to as many times four right angles as there are units in S − 2, S being the number of solid angles of the polyhedron.

The plane angles = sum of all the interior angles of each face, which Prop. 32, of the 1st book of Euclid

$$= \text{H}\,(n - 2)\,.\,\pi$$
$$= 2\,(\text{A} - \text{H})\,\pi\ (\text{since } n\,\text{H} = 2\,\text{A})$$
$$= (\text{S} - 2)\,2\,\pi\ (\text{since A} - \text{H} = \text{S} - 2).$$

51. There can be only five regular polyhedrons.

Since every face has n plane angles, the number of plane angles which compose all the solid angles $= n\text{H} = \text{S}m = 2\,\text{A}$, and by the last article $\text{S} + \text{H} = \text{A} + 2$;

$$\therefore\ \text{H} = \frac{m}{n}\,\text{S}, \quad \text{and} \quad \text{A} = \frac{m\text{S}}{2}; \text{ substituting these,}$$

$$\text{S} + \frac{m}{n}\,\text{S} = \frac{m\,\text{S}}{2} + 2,$$

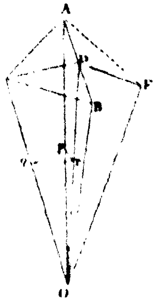
$$2n\text{S} + 2m\text{S} = mn\text{S} + 4n,$$

$$2n\text{S} + 2m\text{S} - mn\text{S} = 4n,$$

$$\text{S}\,\{2\,(n + m) - mn\} = 4n,$$

$$\text{S} = \frac{4n}{2\,(n + m) - mn}.$$

Now this must be a positive whole number, and in order that it may be so $2\,(m + n)$ must be greater than mn; and, therefore, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$, or $\frac{1}{n} > \frac{1}{2} - \frac{1}{m}$; but m cannot be less than 3, therefore, $\frac{1}{n}$ cannot be so small as $\frac{1}{2} - \frac{1}{3}$, or $\frac{1}{6}$, consequently, since n must be an integer and cannot be less than 3, it can only be 3, 4, or 5. In the same manner m cannot be less than 3, therefore the values of m can only be 3, 4, or 5.



52. To find the inclination of two adjacent faces of a polyhedron to each other.

Let AB be the edge common to the two adjacent faces of a polyhedron, C and E the centres of the faces. Draw CO, EO, perpendicular to the faces meeting each other in O; and CD and ED, perpendicular to AB, the intersection of the planes ABC, ABE, then the angle CDE is the required inclination.

Let n be the number of sides in each face, m the number of plane angles in each solid angle if from the centre O, and radius equal to unity, describe a spherical triangle meeting the lines OA, OC, OD in p, q, r, we shall have spherical triangle pqr, in which we have the angle r a right angle, the angle $p = \frac{\pi}{m}$, and the angle $q = \frac{\pi}{n}$;

$$\text{and by right-angled triangles, } \cos qr = \frac{\cos p}{\sin q};$$

but $\cos qr = \cos \text{COD} = \sin \text{CDO} = \sin \frac{1}{2} \text{C}$, C being the angle CDE; then

$$\sin \tfrac{1}{2} \text{C} = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}.$$

This equation is general, and applies successively to the five polyhedrons, by substituting the values of m and n in each case.

Tetrahedron $m = 3$, $n = 3$. **Hexahedron** $m = 3$, $n = 4$.

Octahedron $m = 4$, $n = 3$. Dodecahedron $m = 3$, $n = 5$.

Icosahedron $m = 5$, $n = 3$.

From the triangle pqr from which we have deduced the inclination of the two adjacent faces, we have

$$\cos pq = \cot p \cot q; \text{ or } \frac{\text{CO}}{\text{OA}} = \cot \frac{\pi}{m} \cot \frac{\pi}{n};$$

therefore, if we call R the radius of the sphere which circumscribes the polyhedron, and r the radius of the sphere inscribed in it, we shall have $\frac{R}{r} = \tan\frac{\pi}{m} \tan\frac{\pi}{n}$;

and, by making the side AB $= a$, we have CA $= \frac{\frac{1}{2} a}{\sin\frac{\pi}{n}}$;

$$\text{and, consequently, } R^2 = r^2 + \frac{\frac{1}{4} a^2}{\sin^2\frac{\pi}{n}}.$$

These two equations give for each polyhedron, the values of the radii R and r for the circumscribed and inscribed sphere. We have, supposing C known,

$$r = \tfrac{1}{2} a \cot\frac{\pi}{n} \tan\tfrac{1}{2} C \text{ and } R = \tfrac{1}{2} \tan\frac{\pi}{m} \tan\tfrac{1}{2} C.$$

In the dodecahedron and icosahedron, $\frac{R}{r}$ has the same value for both; viz., $\tan\frac{\pi}{3} \tan\frac{\pi}{5}$. Therefore, if R be the same for both, r will also be the same; that is to say, if these two solids are inscribed in the same sphere, they will also circumscribe the same sphere, and *vice versâ*.

The same property holds with regard to the hexahedron and octahedron, since the value of $\frac{R}{r}$ is the same for one as the other; viz., $\tan\frac{\pi}{3} \tan\frac{\pi}{4}$.

53. To find the inclination of two adjacent faces in the five regular polyhedrons.*

From the equation $\sin\frac{1}{2} C = \frac{\cos\frac{\pi}{m}}{\sin\frac{\pi}{n}}$, taking the tetrahedron where $m = 3$ and $n = 3$,

---

* Legendre, at page 312 of his Geometry, finds the inclination from the equation $\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$; see equation 3, page 5.

$$\sin \tfrac{1}{2}\mathrm{C} = \frac{\cos 60^\circ}{\sin 60^\circ} = \frac{1}{\sqrt{3}}; \therefore \cos \mathrm{C} = \tfrac{1}{3}.$$

In the hexahedron $m = 3$ and $n = 4$,

$$\sin \tfrac{1}{2}\mathrm{C} = \frac{\cos 60}{\sin 45} = \frac{1}{\sqrt{2}}, \text{ and } \cos \mathrm{C} = 0; \quad \therefore \mathrm{C} = 90.$$

In the octahedron $m = 4$ and $n = 3$,

$$\sin \tfrac{1}{2}\mathrm{C} = \frac{\cos 45}{\sin 60} = \sqrt{\frac{2}{3}}, \text{ and } \cos \mathrm{C} = -\tfrac{1}{3}.$$

In the dodecahedron $m = 3$, $n = 5$,

$$\sin \tfrac{1}{2}\mathrm{C} = \frac{\cos 60}{\sin 36} = \frac{2}{\sqrt{10 - 2\sqrt{5}}}, \text{ and } \cos \mathrm{C} = \frac{1 - \sqrt{5}}{5 - \sqrt{5}}$$

In the icosahedron $m = 5$, $n = 3$,

$$\sin \tfrac{1}{2}\mathrm{C} = \frac{\cos 36}{\sin 60} = \frac{1 + \sqrt{5}}{2\sqrt{5}}, \text{ and } \cos \mathrm{C} = -\frac{\sqrt{5}}{3}.$$

54. To find the solid content of a regular polyhedron.

The area of each face $= \frac{n}{4} a^2 \cot \frac{\pi}{n}$; hence the area of the surface of the polyhedron $= \mathrm{H} \,.\, \frac{n}{4} a^2 \cot \frac{\pi}{n}$, and the solid content $= \dfrac{\text{area of the surface} \times \text{by the altitude}}{3} = \tfrac{1}{3}$ of the area of the surface × by radius of inscribed sphere

$$= \tfrac{1}{3} \times \mathrm{H} \,.\, \frac{n}{4} a^2 \cot \frac{\pi}{n} \times r = \frac{n r a^2 \mathrm{H}}{12} \cot \frac{\pi}{n} \;\ldots\ldots\; (1)$$

$$\left(\text{or since } r = \tfrac{1}{2} a \cot \frac{\pi}{n} \,.\, \tan \tfrac{1}{2}\mathrm{C}\right)$$

$$= \frac{n a^3 \mathrm{H}}{12} \cot^2 \frac{\pi}{n} \tan \tfrac{1}{2}\mathrm{C} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2)$$

From either of these equations we can find the solid content.

We shall here use the first. Taking the equations

$$\frac{R}{r} = \tan\frac{\pi}{m}\tan\frac{\pi}{n}, \text{ and } R^2 - r^2 = \frac{a^2}{4\sin^2\frac{\pi}{n}},$$

we can find r and R.

In the tetrahedron $m = 3$, $n = 3$;

$$\therefore \frac{R}{r} = \tan 60 \,.\, \tan 60 = \sqrt{3} \times \sqrt{3} = 3; \quad \therefore R = 3r.$$

$$R^2 - r^2 = \frac{a^2}{4\sin^2 60} = \frac{a^2}{3}, \text{ but } R = 3r.$$

$$\therefore (3r)^2 - r^2 = \frac{a^2}{3} \text{ or } 8r^2 = \frac{a^2}{3};$$

$$\therefore r^2 = \frac{a^2}{24}, \quad r = \frac{a}{2\sqrt{6}}. \qquad \text{Also } \frac{R}{3} = r$$

$$\therefore R^2 - \frac{R^2}{9} = \frac{a^2}{3}. \qquad R = \frac{3a}{2\sqrt{6}}.$$

In the hexahedron $m = 3$, $n = 4$,

$$\frac{R}{r} = \tan 60 \tan 45 = \sqrt{3}; \quad r = \frac{a}{2}; \quad \therefore R = \frac{a\sqrt{3}}{2}.$$

In the octahedron, $m = 4$, and $n = 3$, and $r = \dfrac{a}{\sqrt{6}}$; $R = \dfrac{a}{\sqrt{2}}$.

In the dodecahedron, $m = 3$, $n = 5$,

$$r = \frac{a}{20}\sqrt{250 + 110\sqrt{5}}; \quad R = \frac{a}{4}(\sqrt{15} + \sqrt{3}).$$

In the icosahedron, $m = 5$, $n = 3$,

$$r = \frac{a}{12}\sqrt{42 + 18\sqrt{5}}; \quad R = \frac{a}{4}\sqrt{10 + 2\sqrt{5}}.$$

These being substituted in equation (1), page 65, we find the solidity of each of the five polyhedrons.

For the tetrahedron the solidity is $\frac{a^3}{12}\sqrt{2}$

For the hexahedron .... a^3.

For the octahedron.................. $\frac{a^3}{3}\sqrt{2}$.

For the dodecahedron .... $\frac{a^3}{4}\sqrt{470+210\sqrt{5}}$.

For the icosahedron .... $\frac{5a^3}{12}\sqrt{14+6\sqrt{5}}$

Examples.

1. In the oblique-angled spherical triangle ABC. Given the side AB 73° 13′, the side BC 62° 12′, the side AC 119° 5′, required the angles.

$$\text{Ans.}\begin{cases} A = 44°\ 18′. \\ B = 136°\ 40′. \\ C = 48°\ 48′. \end{cases}$$

2. The latitudes and longitudes of three places on the earth's surface, suppose London, Moscow, Constantinople, being given as below: required the latitude and longitude of that place which is equidistant from the former three?

The latitude of London is 51° 30′, the latitude and longitude of Moscow 55° 45′, and 38°, and those of Constantinople 41° 30′ and 29° 15′ respectively.

3. Given the latitude of three places, Moscow 55° 30′, Vienna 48° 12′, Gibraltar 35° 30′, all lying directly in the same arc of a great circle. The difference of longitude between Vienna, (situated in the middle,) and Moscow, easterly, is equal to that between Vienna and Gibraltar, westerly. It is required to find the true bearing and distance of each place from the other, and the difference of longitude, according to the convexity of the globe.

4. Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space inclosed between them and the three triangular planes through each three centres.

5. A point P being taken in the surface of a sphere, let α, β denote its spherical distances from two given points; then if $m \cos \alpha + n \cos \beta =$ a constant quantity, m and n being any given numbers, the locus of P will be a circle.

6. The base of a spherical triangle is given, and the sum of the cosines of the angles at the base, to trace the locus of its vertex.

7. The sides of a spherical triangle are produced to meet again in three more points, thus forming, with the original, four spherical triangles, which constitute Davies's "Associated Triangles;" (12th edit., Hutton's Course, vol. ii. p. 41,) $r\ r_1\ r_2\ r_3$ are the radii of the inscribed, and $R\ R_1\ R_2\ R_3$ the radii of the circumscribed circles. Prove that

$$\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3 =$$
$$\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3.$$

8. A person engages to travel from London to Constantinople, and to touch the equator in his journey, required the point of contact, and the length of his track, admitting it to be the shortest possible, and the earth a sphere.

9. The angular points of two triangular pyramids being respectively situated on four converging lines in space, let the corresponding faces be produced to meet; then will the four lines of section be all situated in the same plane.

10. Given the longitudes of two places, 6° 49′, and 54° 35′, their respective latitudes 48° 23′ 14″, and 4° 56′ 15″; find their distance, the longitudes being both west, and their latitudes both north.

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